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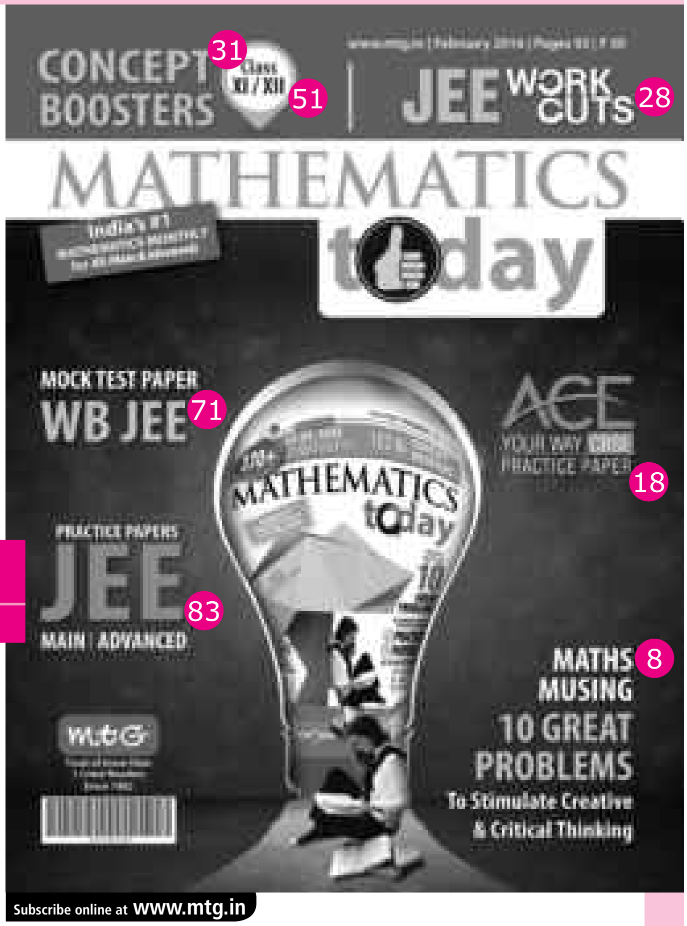

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Editor : Anil Ahlawat

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 158

JEE MAIN

1. Let a, b, c be an increasing sequence of positive integers in which $a = 2010$ and b divides c. If N is the number of such sequences where a, b, c are in H.P., the last digit of N is

(a) 3 (b) 5 (c) 6 (d) 7

2. If $2 \le |z| \le 4$ and $a \le \left|z + \frac{1}{z}\right| \le b$, then $a + b =$

(a) $\frac{19}{4}$ (b) $\frac{21}{4}$ (c) $\frac{23}{4}$ (d) $\frac{21}{5}$

3. The alternate vertices of a regular octagon are joined to form another octagon. The ratio of the areas of the two octagons is

(a) $\sqrt{2}-1$ (b) $2-\sqrt{2}$

(c) $3-2\sqrt{2}$ (d) $\sqrt{3}-1$

4. The determinant

$$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix} = 0$$

for

(a) $a + b + c + d = 0$

(b) $ab + cd = 0$

(c) $ab\,(c + d) + cd\,(a + b)$

(d) any a, b, c, d

5. $ABCD$ is a rectangle with A $(-1, 2)$, B $(3, 7)$ and $AB : BC = 4 : 3$. If d is the distance of the origin from the centre of the rectangle, then $[d]$ =

(a) 4 (b) 5 (c) 6 (d) 7

JEE ADVANCED

6. Let $x \in \left(0, \frac{\pi}{2}\right)$, $f(x) = x\tan(\sin x)$, $g(x) = x\sin(\tan x)$ and $h(x) = \sin x \tan x$. Which one is greatest?

(a) $f(x)$ (b) $g(x)$ (c) $h(x)$ (d) depends on x

COMPREHENSION

Let $f(x) = \dfrac{(x-1)(x-2)}{(x-a)(x-b)}$

7. If $a < b < 1$, then $f(x)$ has

(a) neither maximum nor minimum

(b) a maximum

(c) a minimum

(d) a maximum and a minimum

8. If $1 < a < b < 2$, then $f(x)$ has

(a) neither maximum nor minimum

(b) a maximum

(c) a minimum

(d) a maximum and a minimum

INTEGER MATCH

9. In a triangle ABC, if the angles are in the ratio $1 : 2 : 4$, then $\dfrac{\sum \sec^2 A}{\sum \operatorname{cosec}^2 A} = \ldots\ldots$

MATCHING LIST

10. Match the following:

(a)	If 4 dice are rolled, the probability of getting the sum 10, is	(p)	$\frac{7}{15}$
(b)	If x and y are selected from the set of the first 10 natural numbers, the probability that $x^2 - y^2$ is divisible by 3, is	(q)	$\frac{2}{7}$
(c)	If 10 men are sitting in a row, the probability of choosing 3 of them so that no two are from adjacent seats is	(r)	$\frac{8}{15}$
(d)	Triangles are formed with vertices of a regular octagon. If a triangle is chosen at random, the probability that it does not have any side common with the octagon, is	(s)	$\frac{5}{27}$
		(t)	$\frac{5}{81}$

See Solution set of Maths Musing 157 on page no. 81

PRACTICE PAPER 2016
JEE MAIN

1. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. Let $t_n = \alpha^n - \beta^n$ for $n \in N$, then the value of $\frac{t_{22} - 2t_{20}}{2t_{21}}$ is
(a) 1 (b) 2 (c) 3 (d) 4

2. Let z be a complex number such that the imaginary part of z is non–zero and $a = z^2 + z + 1$ is real. Then a cannot take the value
(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

3. Let $a_1, a_2, a_3, \ldots\ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. If n is the least positive integer for which $a_n < 0$ then the value of $4n - 100$ is
(a) 0 (b) 1 (c) 2 (d) 3

4. Six cards and six envelopes are numbered 1, 2, 3, 4, 5 and 6. Cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is
(a) 264 (b) 265 (c) 53 (d) 67

5. Let $A = \{\alpha : \sin\alpha - \cos\alpha = \sqrt{2}\cos\alpha\}$ and $B = \{\alpha : \sin\alpha + \cos\alpha = \sqrt{2}\sin\alpha\}$ be two sets. Then
(a) $A \subset B$ and $B - A \neq \phi$ (b) $B \not\subset A$
(c) $A \not\subset B$ (d) $A = B$

6. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively. If $x = \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$, then x is root of the equation
(a) $x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$
(b) $x^2 - (\sqrt{2}+1)x + \sqrt{2} = 0$
(c) $x^2 - (\sqrt{3}+\sqrt{2})x + \sqrt{5} = 0$
(d) None of these

7. Tangents drawn from the point $M(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at points A and B. If C is the centre of the circle. Then the equation of circumcircle of triangle ABC is
(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 20 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

8. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{3}{r}$ (d) $\pm\frac{2}{r}$

9. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinates axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is
(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

10. If $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1}x)$, then the value of $2x + 3$ is
(a) 0 (b) 1 (c) 2 (d) 3

11. If A is an 3×3 non–singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then $(BB')^{10}$ equals
(a) $I + B$ (b) I (c) B^{-1} (d) $(B^{-1})'$

Contributed by : Sanjay Singh, Mathematics Classes, Chandigarh

12. The sum of values of λ for which the system of linear equations

$2x_1 - 2x_2 + x_3 = \lambda x_1$
$2x_1 - 3x_2 + 2x_3 = \lambda x_2$
$-x_1 + 2x_2 = \lambda x_3$

has a non–trivial solution is

(a) 2 (b) –2
(c) 4 (d) None of these

13. Four fair dice D_1, D_2, D_3, D_4 each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{125}{216}$ (d) $\frac{127}{216}$

14. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[x]$ denotes the greatest integer $\le x$) has no integral solution, then all possible values of a lie in the interval

(a) $(-1, 0) \cup (0, 1)$ (b) $(1, 2)$
(c) $(-2, -1)$ (d) $(-\infty, -2) \cup (2, \infty)$

15. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then value of $2a - b$ is

(a) 2 (b) 4
(c) 6 (d) None of these

16. If $f(x) : [1, 10] \to Q$ be a continuous function. If $f(x)$ takes rational value for all x and $f(2) = 5$ then the equation whose roots are $f(3)$ and $f(\sqrt{10})$ is

(a) $x^2 - 10x + 25 = 0$ (b) $x^2 - 3x + 2 = 0$
(c) $x^2 - 6x + 5 = 0$ (d) None of these

17. If $f''(x) = -f(x)$ and $g(x) = f'(x)$.

If $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then value of the expression $(F(10))^3 - 5(F(5))^2 + F(2)$ is

(a) 5 (b) 10 (c) 0 (d) 15

18. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1) meets the curve again at (a, b) then the value of $(2a - 3b)$ is

(a) 8 (b) 9
(c) 10 (d) None of these

19. The number of roots of the equation $x^2 - x\sin x - \cos x = 0$ is

(a) 6 (b) 4 (c) 2 (d) 0

20. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}dx$ is

(a) $\frac{1}{4}\ln\frac{3}{2}$ (b) $\frac{1}{2}\ln\frac{3}{2}$
(c) $\ln\frac{3}{2}$ (d) $\frac{1}{6}\ln\frac{3}{2}$

21. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $[0, \pi/2]$ is given as $2\sqrt{a}(\sqrt{b} - c)$ where a and b are prime number then the value of a, b and c respectively.

(a) (2, 2, 1) (b) (2, 2, –1)
(c) (3, 2, –1) (d) None of these

22. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

(a) $\frac{1}{3x} + \frac{2x^2}{3}$ (b) $-\frac{1}{3x} + \frac{4x^2}{3}$
(c) $-\frac{1}{x} + \frac{2}{x^2}$ (d) $\frac{1}{x}$

23. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$, respectively. The quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither a rhombus nor a rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

24. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is

(a) $5x - 11y + z = 17$ (b) $\sqrt{2}x + y = 3\sqrt{2} - 1$
(c) $x + y + z = \sqrt{3}$ (d) $x - \sqrt{2}y = 1 - \sqrt{2}$

25. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(a) $x + 2y - 2z = 0$ (b) $3x + 3y - 2z = 0$
(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

26. If $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors, then $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$ does not exceed

(a) 4 (b) 9 (c) 8 (d) 6

27. Let O (0, 0), P (3, 4) and Q (6, 0) be the vertices of triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

(a) $\left(\frac{4}{3}, 3\right)$ (b) $\left(3, \frac{2}{3}\right)$ (c) $\left(3, \frac{4}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{2}{3}\right)$

28. The tangents of angles subtended by a tower at four points A, B, C and D on the ground are in H.P. If O be the foot of the tower on the ground, then

(a) $OA + OC = OB + OD$
(b) $OA + OB = OC + OD$
(c) $OA + OD = OB + OC$
(d) $AB + CD = BC + CD$

29. Statement – 1 : $e^{\pi} > \pi^{e}$

Statement – 2 : The function $x^{1/x}$ $(x > 0)$ has local maximum at $x = e$.

(a) Statement I is true, Statement II is true ; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true ; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

30. Statement I : $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

Statement II : $\sim(p \leftrightarrow \sim q)$ is a tautology.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true; Statement I is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

SOLUTIONS

1. (c) : $t_n = \alpha^n - \beta^n$

Also $\alpha^2 - 6\alpha - 2 = 0$

Multiply with α^{20} on both sides

$\Rightarrow \alpha^{22} - 6\alpha^{21} - 2\alpha^{20} = 0$ ... (1)

Similarly, $\beta^{22} - 6\beta^{21} - 2\beta^{20} = 0$ ... (2)

Subtracting (2) from (1) we have

$$\alpha^{22} - \beta^{22} - 6(\alpha^{21} - \beta^{21}) = 2(\alpha^{20} - \beta^{20})$$

$$\Rightarrow t_{22} - 6t_{21} = 2t_{20} \Rightarrow \frac{t_{22} - 2t_{20}}{2t_{21}} = 3$$

2. (d) : Given equation is $z^2 + z + 1 - a = 0$

Clearly this equation do not have real roots if $D < 0$

$\Rightarrow 1 - 4(1 - a) < 0 \Rightarrow 4a < 3 \Rightarrow a < \frac{3}{4}$

3. (a) : $a_1, a_2, a_3, \ldots$.. are in H.P.

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$.. are in A.P.

Let the common difference of A.P. be d, then

$$\frac{1}{a_{20}} = \frac{1}{a_1} + (20-1)d$$

$$\Rightarrow \frac{1}{a_{20}} - \frac{1}{a_1} = 19d \Rightarrow \frac{1}{25} - \frac{1}{5} = 19d$$

$$\Rightarrow d = \left(\frac{-4}{19 \times 25}\right) \Rightarrow \frac{1}{a_n} = \frac{1}{a_1} + (n-1)d < 0$$

$$\Rightarrow \frac{1}{5} + (n-1)\left(\frac{-4}{19 \times 25}\right) < 0$$

or $\frac{4(n-1)}{19 \times 5} > 1$ or $n - 1 > \frac{19 \times 5}{4}$

or $n > \frac{19 \times 5}{4} + 1$ or $n \geq 25$.

Least positive value of $n = 25$

Hence the value of $4n - 100 = 4 \times 25 - 100 = 0$

4. (c) : There are 2 possibilities

(i) If the card number '2' goes in the envelope '1' then it is derangement of 4 things which can be done in $4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$ ways.

(ii) If card number '2' doesn't go in the envelope '1', then it is derangement of 5 things which can be done in $5!\left(\frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!}\right) = 44$ ways.

Hence, total 53 ways are there.

5. (d) : In set A, $\sin\alpha = (\sqrt{2}+1)\cos\alpha$

or $\tan\alpha = \sqrt{2} + 1$

In set B, $(\sqrt{2}-1)\sin\alpha = \cos\alpha$

or $\tan\alpha = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \quad \therefore \; A = B$

6. (a) : Since angles of ΔABC are in A.P., $2B = A + C$

Also, $A + B + C = 180°$

$\therefore \; B = 60°$

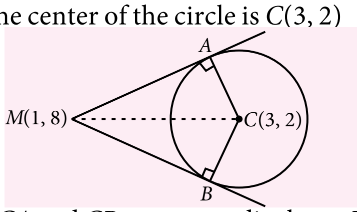
$$\therefore \; x = \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = 2\sin C\cos A + 2\sin A\cos C$$

$$= 2\sin(A+C) = 2\sin 2B = 2\frac{\sqrt{3}}{2} = \sqrt{3}$$

Clearly x is a root of the equation

$x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$

7. **(b):** The center of the circle is $C(3, 2)$

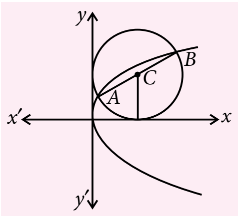


Since, CA and CB are perpendicular to MA and MB, CM is the diameter of the circumcircle of ΔABC.
Its equation is

$$(x-3)(x-1)+(y-2)(y-8)=0$$

or $x^2+y^2-4x-10y+19=0$

8. **(d):**



We have points $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ on the parabola $y^2 = 4x$

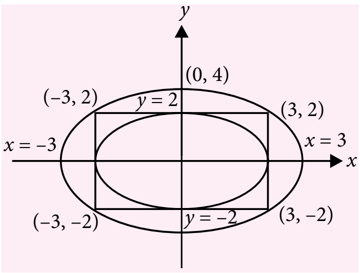
For circle on AB as diameter center is $C\left(\frac{t_1^2+t_2^2}{2}, (t_1+t_2)\right)$.

Since circle is touching the x–axis, we have

$$r = |t_1+t_2| \text{ or } t_1+t_2 = \pm r$$

Also slope of AB, $m = \frac{2t_1-2t_2}{t_1^2-t_2^2} = \frac{2}{t_1+t_2} = \pm\frac{2}{r}$

9. **(c) :** Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ as it is passing through (0, 4) and (3, 2).



So, $b^2 = 16$ and $\frac{9}{a^2}+\frac{4}{16} = 1$

or $a^2 = 12$

So, $12 = 16(1-e^2)$ or $e = \frac{1}{2}$

10. **(c) :** $\sin[\cot^{-1}(x+1)]$

$$= \sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right) = \frac{1}{\sqrt{x^2+2x+2}}$$

$$\cos(\tan^{-1}x) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

Thus, $\frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$

or $x^2+2x+2 = 1+x^2$ or $x = -\frac{1}{2}$

Hence the value of $2x+3 = 2$.

11. **(b):** $B = A^{-1}A' \Rightarrow AB = A'$

$\Rightarrow ABB' = A'B' = (BA)' = (A^{-1}A'A)' = (A^{-1}AA')' = A$

$\Rightarrow BB' = I$

Hence $(BB')^{10} = I^{10} = I$

12. **(b):** System has non–trivial solution

$$\therefore \begin{vmatrix} \lambda-2 & 2 & -1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda-2)(3\lambda+\lambda^2-4)-2(-2\lambda+2)-1(4-3-\lambda) = 0$

$\Rightarrow (\lambda-2)(\lambda-1)(\lambda+4)+4(\lambda-1)+(\lambda-1) = 0$

$\Rightarrow (\lambda-1)(\lambda^2+2\lambda-8+5) = 0 \Rightarrow (\lambda-1)(\lambda^2+2\lambda-3) = 0$

$\Rightarrow (\lambda-1)^2(\lambda+3) = 0 \Rightarrow \lambda = 1, -3$

Hence the sum is $1+(-3) = -2$

13. **(a) :** Required probability

$= 1 - P$ (Dice D_4 shows none of D_1, D_2 and D_3)

D_4 can show any of six numbers.

If D_4 shows a number x, then D_1, D_2, D_3 shows any number other than x.

$$\therefore \text{ Required probability} = 1 - {}^6C_1 \times \frac{1}{6}\times\left(\frac{5}{6}\right)^3$$

$$= 1-\frac{6\cdot 5^3}{6^4} = 1-\frac{125}{216} = \frac{91}{216}$$

14. **(a) :** Let $t = x-[x] = \{x\}$

$a^2 = 3t^2-2t$, where $t = \{x\} \in (0, 1)$

$$= 3\left[t^2-\frac{2}{3}t\right] = 3\left[t^2-2t.\frac{1}{3}+\frac{1}{9}-\frac{1}{9}\right]$$

$$= 3\left[\left(t-\frac{1}{3}\right)^2-\frac{1}{9}\right] = 3\left(t-\frac{1}{3}\right)^2-\frac{1}{3}$$

Since $0 < t < 1$

$$-\frac{1}{3} < t-\frac{1}{3} < \frac{2}{3}$$

$\Rightarrow 0 \le \left(t-\frac{1}{3}\right)^2 < \frac{4}{9} \Rightarrow 0 \le 3\left(t-\frac{1}{3}\right)^2 < \frac{4}{3}$

$\Rightarrow -\frac{1}{3} \le 3\left(t-\frac{1}{3}\right)^2-\frac{1}{3} < 1 \Rightarrow -\frac{1}{3} \le a^2 < 1$

$\Rightarrow 0 < a^2 < 1$

or $a \in (-1, 0) \cup (0, 1)$

15. (c) : $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$

$\Rightarrow \lim_{x\to\infty}\left(x+\frac{1}{x+1}-ax-b\right)=4$

$\Rightarrow \lim_{x\to\infty}((1-a)x-b)=4$

This is possible only when $(1 - a) = 0$ and $-b = 4$

$\Rightarrow$ $a = 1$ and $b = -4$

Hence $2a - b = 6$

16. (a) : Since $f(x)$ is continuous in [1, 10].

$f(x)$ will attain all values between $f(1)$ and $f(10)$. If $f(1) \neq f(10)$, then $f(x)$ will attain innumerable irrational values between $f(1)$ and $f(10)$. But given that $f(x)$ attains rational values only, then we must have $f(1) = f(10)$, infact $f(x)$ = constant for $x \in [1, 10]$. Since $f(2) = 5$, $f(x) = f(2) = 5$, $\forall\ x$.

Hence the equation whose roots are $f(3)$ and $f(\sqrt{10})$ is $x^2 - (5 + 5)x + 25 = 0$.

17. (a) : Given $f''(x) = -f(x)$ and $g(x) = f'(x)$

$F(x)=\left(f\left(\frac{x}{2}\right)\right)^2+\left(g\left(\frac{x}{2}\right)\right)^2$

$\therefore\ F'(x)=f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)$

$=f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)f''\left(\frac{x}{2}\right)$

$=f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)-g\left(\frac{x}{2}\right)f\left(\frac{x}{2}\right)=0$

Thus $F(x) = c$

Hence $F(x) = F(5) = 5 = F(10) = F(2)$

18. (b): $x^2 + 2xy - 3y^2 = 0$

$(x - y)(x + 3y) = 0$

$x - y = 0$ or $x + 3y = 0$

Equation of normal at (1, 1) to line $x-y=0$ is $x+y-2=0$, which intersects the line $x + 3y = 0$ at (3, –1).

Hence the value of $(2a - 3b) = 2(3) - 3(-1) = 9$

19. (c) : Let $f(x) = x^2 - x\sin x - \cos x$

$\therefore\ f'(x) = 2x - x\cos x = x(2 - \cos x)$

Clearly $f(x)$ is decreasing in $(-\infty, 0)$, increasing in $(0, \infty)$.

Hence $x = 0$ is point of minima.

$f(0)=-1<0$ and $\lim_{x\to\infty} f(x)\to\infty,\ \lim_{x\to-\infty} f(x)\to\infty$

Therefore, it meets x–axis at two points.

Hence number of roots is 2.

20. (a) : Put $x^2 = t$ or $2x\,dx = dt$

$$\therefore\ I=\frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin t}{\sin t+\sin(\ln 6-t)}dt \quad ...(1)$$

$$\therefore\ I=\frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin(\ln 6-t)}{\sin(\ln 6-t)+\sin t}dt \quad ...(2)$$

Using

$\left(\int_a^b f(x)=dx=\int_a^b f(a+b-x)dx \text{ and } \ln 2+\ln 3=\ln 6\right)$

Adding (1) and (2)

$2I=\frac{1}{2}\int_{\ln 3}^{\ln 3}1\,dt$ or $I=\frac{1}{4}\ln\frac{3}{2}$

21. (a) : Since $\sin x$ and $\cos x > 0$ for $x\in\left[0,\frac{\pi}{2}\right]$, the graph of $y = \sin x + \cos x$ always lies above the graph of $y = |\cos x - \sin x|$. Also $\cos x > \sin x$ for $x\in\left[0,\frac{\pi}{4}\right]$ and $\sin x > \cos x$ for $x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$.

$\Rightarrow$ Area $=\int_0^{\pi/4}((\sin x+\cos x)-(\cos x-\sin x))\,dx+\int_{\pi/4}^{\pi/2}((\sin x+\cos x)-(\sin x-\cos x))\,dx$

$=2\int_0^{\pi/4}\sin x\,dx+2\int_{\pi/4}^{\pi/2}\cos x\,dx$

$=[-2\cos x]_0^{\pi/4}+[2\sin x]_{\pi/4}^{\pi/2}$

$=-\left[\sqrt{2}-2\right]+\left[2-\sqrt{2}\right]=4-2\sqrt{2}=2\sqrt{2}(\sqrt{2}-1)$

Hence by comparison a, b and c are respectively (2, 2, 1)

22. (a) : $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1 \qquad \left(\frac{0}{0}\text{form}\right)$

$\Rightarrow \lim_{t\to x}\frac{2tf(x)-x^2f'(t)}{1}=1$

(Applying L' Hospital's Rule)

$\Rightarrow x^2f'(x) - 2xf(x) + 1 = 0$

$\Rightarrow \frac{x^2f'(x)-2xf(x)}{x^4}=-\frac{1}{x^4}$

$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^2}\right)=-\frac{1}{x^4}$

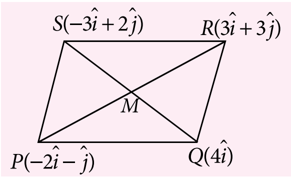
$\Rightarrow \int d\left(\frac{f(x)}{x^2}\right)=\int-\frac{1}{x^4}dx \Rightarrow \frac{f(x)}{x^2}=\frac{1}{3x^3}+c$

Since, $f(1)=1\Rightarrow c=\frac{2}{3}$

$\Rightarrow \frac{f(x)}{x^2}=\frac{1}{3x^3}+\frac{2}{3}\Rightarrow f(x)=\frac{1}{3x}+\frac{2x^2}{3}$

23. (a) : Evaluating midpoint of PR and QS which gives

$M=\left[\frac{\hat{i}}{2}+\hat{j}\right]$, same for both.



$\overrightarrow{PQ}=\overrightarrow{SR}=6\hat{i}+\hat{j}$

$\overrightarrow{PS}=\overrightarrow{QR}=-\hat{i}+3\hat{j}$

So, $\overrightarrow{PQ}\cdot\overrightarrow{PS}\neq 0$

$\overrightarrow{PQ}\,||\,\overrightarrow{SR}$, $\overrightarrow{PS}\,||\,\overrightarrow{QR}$ and $|\overrightarrow{PQ}|\,|\overrightarrow{SR}|=\sqrt{37}$,

$|\overrightarrow{PS}|=|\overrightarrow{QR}|=\sqrt{10}$

Hence, $PQRS$ is a parallelogram but neither a rhombus nor a rectangle.

24. (a) : Equation of required plane is

$P\equiv(x+2y+3z-2)+\lambda(x-y+z-3)=0$

$\Rightarrow\ (1+\lambda)x+(2-\lambda)y+(3+\lambda)z-(2+3\lambda)=0$

Its distance from $(3, 1, -1)$ is $\frac{2}{\sqrt{3}}$

$$\Rightarrow\ \frac{2}{\sqrt{3}}=\frac{|3(1+\lambda)+(2-\lambda)-(3+\lambda)-(2+3\lambda)|}{\sqrt{(\lambda+1)^2+(2-\lambda)^2+(3+\lambda)^2}}$$

$$\Rightarrow\ \frac{4}{3}=\frac{(-2\lambda)^2}{3\lambda^2+4\lambda+14}\Rightarrow 3\lambda^2+4\lambda+14=3\lambda^2$$

$$\Rightarrow\ \lambda=-\frac{7}{2}$$

Required plane is $-\frac{5}{2}x+\frac{11}{2}y-\frac{z}{2}+\frac{17}{2}=0$

or $\ 5x-11y+z=17$

25. (c) : Let the planes containing lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ be P_1.

Vector normal to plane P_1 is $\vec{n}_1=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 2\\ 4 & 2 & 3\end{vmatrix}$

$=8\hat{i}-\hat{j}-10\hat{k}$

Now, required plane P_2 is perpendicular to this plane and it contains the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$.

Since all lines pass through the origin, plane P_2 passes through the origin.

Let's find vector $\vec{n}_2$ normal to plane P_2.

$\vec{n}_2=\vec{n}_1\times(2\hat{i}+3\hat{j}+4\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 8 & -1 & -10\\ 2 & 3 & 4\end{vmatrix}=26\hat{i}-52\hat{j}+26\hat{k}$$

Thus equation of plane P_2 having normal $\vec{n}_2$ and passing through $(0, 0, 0)$ is

$26x-52y+26z=0$ or $x-2y+z=0$

26. (b) : $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors.

Now, $x=|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$

$=2(\hat{a}\cdot\hat{a}+\hat{b}\cdot\hat{b}+\hat{c}\cdot\hat{c})-2\hat{a}\cdot\hat{b}-2\hat{b}\cdot\hat{c}-2\hat{c}\cdot\hat{a}$

$=6-2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})$ ... (1)

Also, $|\hat{a}+\hat{b}+\hat{c}|\geq 0$

$\therefore\ \ \hat{a}\cdot\hat{a}+\hat{b}\cdot\hat{b}+\hat{c}\cdot\hat{c}+2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})\geq 0$

or $\ 3+2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})\geq 0$

or $\ 2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})\geq -3$

or $\ -2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})\leq 3$

or $\ 6-2(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a})\leq 9$ ... (2)

From (1) and (2), $x\leq 9$

Therefore, x does not exceed 9.

27. (c) : Centroid of triangle divides triangle into three triangles of equal area. So, in given question R is centroid of triangle OPQ, which is given by

$$R\left(\frac{3+6+0}{3},\frac{4+0+0}{3}\right)\text{ or }R\left(3,\frac{4}{3}\right)$$

28. (c) : If α, β, γ δ be the four angles such that $\tan\alpha$, $\tan\beta$, $\tan\gamma$, $\tan\delta$ are in H.P., then $\cot\alpha$, $\cot\beta$, $\cot\gamma$, $\cot\delta$ are in A.P.

$\Rightarrow\ h\cot\alpha$, $h\cot\beta$, $h\cot\gamma$, $h\cot\delta$ are in A.P.

[h is height of tower]

$\Rightarrow\ OA$, OB, OC, OD are in A.P.

$\Rightarrow\ OA+OD=OB+OC$

29. (d): Consider $f(x)=x^{1/x}$

$$f'(x)=x^{1/x}\left(\frac{1-\ln x}{x^2}\right)$$

$f(x)$ increases on $(0, e)$ and decreases on (e, ∞)

f has local maximum at $x=e$

Since $\pi>e$ and f decreases on (e, ∞) so $f(\pi)<f(e)$

$\Rightarrow\ \pi^e>e^\pi$

30. (c) :

p	q	$\sim q$	$p\leftrightarrow\sim q$	$\sim(p\leftrightarrow\sim q)$	$p\leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

From the last two columns, we get

$\sim(p\leftrightarrow(\sim q))\equiv p\leftrightarrow q$

$\therefore$ Statement I is true.

But $\sim(p\leftrightarrow\sim q)$ is not a tautology, so Statement II is false.

ACE YOUR WAY CBSE XII

PRACTICE PAPER 2016

Time Allowed : 3 hours | Maximum Marks : 100

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) Please check that this Question Paper contains 26 Questions.
(iii) Marks for each question are indicated against it.
(iv) Questions 1 to 6 in Section-A are Very Short Answer Type Questions carrying **one mark** each.
(v) Questions 7 to 19 in Section-B are Long Answer I Type Questions carrying **4 marks** each.
(vi) Questions 20 to 26 in Section-C are Long Answer II Type Questions carrying **6 marks** each.
(vii) Please write down the serial number of the Question before attempting it.

SECTION-A

1. If A is a square matrix satisfying $A^2 = I$, then what is the inverse of A?

2. Show that the points $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$ are collinear.

3. Equation of the plane passing through $(2, 3, -1)$ and is perpendicular to the vector $3\hat{i}-4\hat{j}+7\hat{k}$.

4. Determine the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$.

5. Evaluate : $\int_{-\pi/2}^{\pi/2} \sin x\cos^2 x(\sin^2 x+\cos x)\,dx$

6. Evaluate : $\int\frac{dx}{\sqrt{9-25x^2}}$

SECTION-B

7. Find $\frac{dy}{dx}$, when $y=\sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$, where a is a constant.

8. Differentiate the function with respect to x:

$$\log\left(\frac{a+b\sin x}{a-b\sin x}\right)$$

9. Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards, if the card drawn is not replaced after the first draw.

OR

A clever student used a biased coin so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution and mean of numbers of tails. Is this a good tendency? Justify your answer.

10. Prove that :

$$2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right)=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}\right)$$

11. Find the interval in which the value of the determinant of the matrix A lies.

$$\text{Given } A=\begin{bmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{bmatrix}$$

12. Prove that : $\int_0^{\pi} x\sin^3 x\,dx = \frac{2\pi}{3}.$

OR

Evaluate : $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)dx$

13. Find the length and the coordinates of the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.

14. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation.

OR

Let '$*$' be a binary operation on the set {0, 1, 2, 3, 4, 5}

and $a * b = \begin{cases} a+b, \text{ if } a+b<6 \\ a+b-6, \text{ if } a+b \geq 6 \end{cases}$

Find the identity element and the inverse element of each element of the set for the operation '$*$'.

15. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, then find A^{-1}, using elementary row operations.

16. If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}, \vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ and $\vec{a} \neq \vec{0}$, then prove that $\vec{b} = \vec{c}$.

17. Evaluate $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx.$

18. Determine the values of a, b and c for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x<0 \\ c, & x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}}, & x>0 \end{cases}$$

may be continuous at $x = 0$.

OR

It is given that for the function $f(x) = x^3 + bx^2 + ax + 5$ on [1, 3], Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b.

19. Solve the differential equation :

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

SECTION-C

20. Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in a sphere of radius R.

21. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the number of hearts in the three cards drawn. Find the mean and variance of X.

OR

Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more beneficial for the patient.

22. A factory owner wants to purchase two types of machines A and B, for his factory. The machine A requires an area of 1000 m^2 and 12 skilled men for running it and its daily output is 50 units, whereas the machine B requires 1200 m^2 area and 8 skilled men, and its daily output is 40 units. An area of 7600 m^2 and 72 skilled men be available to operate the machines.

(i) How many machines of each type should be bought to maximize the daily output?

(ii) Write two advantages of keeping skilled men in a firm.

23. Show that the lines $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ are coplanar. Also, find the equation of the plane containing these lines.

24. Let $f : N \to R$ be a function defined as

$f(x) = 4x^2 + 12x + 15.$

Show that $f : N \to S$ is invertible, where S is the range of f. Also, find inverse of f.

25. Find $\int \frac{x^4}{(x-1)(x^2+1)}dx$

OR

Prove that the curves $y = x^2$ and $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 1$ into three parts which are equal in area.

26. Find the particular solution of the differential equation

$$xe^{\frac{y}{x}} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0, \text{ given that } y = 0,$$

when $x = 1$.

SOLUTIONS

1. We have $A^2 = I \Rightarrow A^{-1}A^2 = A^{-1}I \Rightarrow A = A^{-1}$

2. Position vector of $A = 3\hat{i} - 5\hat{j} + 1\hat{k}$

Position vector of $B = -1\hat{i} + 0\hat{j} + 8\hat{k}$

Position vector of $C = 7\hat{i} - 10\hat{j} - 6\hat{k}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= -4\hat{i} + 5\hat{j} + 7\hat{k}$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$= 4\hat{i} - 5\hat{j} - 7\hat{k}$

$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AB}$

$\therefore$ The points A, B and C are collinear.

3. The equation of the plane passing through (2, 3, – 1) and perpendicular to the vector $3\hat{i} - 4\hat{j} + 7\hat{k}$ is

$3(x - 2) + (-4)(y - 3) + 7(z - (-1)) = 0$

or, $3x - 4y + 7z + 13 = 0$

4. The given equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Clearly, it is an equation of order 2 and degree 2.

5. $\int_{-\pi/2}^{\pi/2} \sin x\cos^2 x(\sin^2 x + \cos x)dx$

$$= \int_{-\pi/2}^{\pi/2} \sin^3 x\cos^2 x\,dx + \int_{-\pi/2}^{\pi/2} \sin x\cos^3 x\,dx = 0$$

($\because$ $\sin x$ is an odd function)

$\therefore$ $\sin^3 x\cos^2 x$ and $\sin x\cos^3 x$ are odd functions.

6. Let $I = \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}}dx$

$$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

7. We have, $y = \sqrt{a + \sqrt{a + \sqrt{a + x^2}}}$, where a is a constant.

$$\Rightarrow y = \left[a + \sqrt{a + \sqrt{a + x^2}}\right]^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a + \sqrt{a + x^2}}\right]^{-\frac{1}{2}} \frac{d}{dx}\left[a + \sqrt{a + \sqrt{a + x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a + \sqrt{a + x^2}}\right]^{-\frac{1}{2}} \left[\frac{1}{2}\left(a + \sqrt{a + x^2}\right)^{-\frac{1}{2}}\right] \times \frac{d}{dx}\left(a + \sqrt{a + x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a + \sqrt{a + x^2}}\right]^{-\frac{1}{2}} \left[\frac{1}{2}\left(a + \sqrt{a + x^2}\right)^{-\frac{1}{2}}\right] \times \left[\frac{1}{2}(a + x^2)^{-\frac{1}{2}}.2x\right]$$

$$\frac{dy}{dx} = \frac{1}{4}x\left[\left(a + \sqrt{a + \sqrt{a + x^2}}\right)\cdot\left(a + \sqrt{a + x^2}\right)\cdot(a + x^2)\right]^{-\frac{1}{2}}$$

8. Let $y = \log\left(\frac{a + b\sin x}{a - b\sin x}\right)$.

Then, $y = \log(a + b\sin x) - \log(a - b\sin x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a + b\sin x} \times \frac{d}{dx}(a + b\sin x) - \left\{\frac{1}{a - b\sin x} \times \frac{d}{dx}(a - b\sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a + b\sin x}(0 + b\cos x) - \frac{1}{a - b\sin x}(0 - b\cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{b\cos x}{a + b\sin x} + \frac{b\cos x}{a - b\sin x}$$

$$\Rightarrow \frac{dy}{dx} = b\cos x\left\{\frac{1}{a + b\sin x} + \frac{1}{a - b\sin x}\right\}$$

$$\Rightarrow \frac{dy}{dx} = b\cos x\left\{\frac{a - b\sin x + a + b\sin x}{(a + b\sin x)(a - b\sin x)}\right\}$$

$$= \frac{2ab\cos x}{a^2 - b^2\sin^2 x}$$

9. Let A be the event of drawing a diamond card in the first draw and B be the event of drawing a diamond card in the second draw. Then,

$$P(A)=\frac{{}^{13}C_1}{{}^{52}C_1}=\frac{13}{52}=\frac{1}{4}$$

After drawing a diamond card in first draw, 51 cards are left out of which 12 cards are diamond cards. $\therefore$ $P(B/A)$ = Probability of drawing a diamond card in second draw when a diamond card has already been drawn in first draw

$$\Rightarrow \quad P(B|A)=\frac{{}^{12}C_1}{{}^{51}C_1}=\frac{12}{51}=\frac{4}{17}$$

Now, req. prob. $= P(A\cap B) = P(B|A)\times P(A)$

$$=\frac{4}{17}\times\frac{1}{4}=\frac{1}{17}$$

OR

$P(H) = 3/4$; $P(T) = 1/4$;

Event: Tails; $n = 2$; $x = 0, 1, 2$

$$P(0)=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16},\quad P(1)=\frac{3}{4}\times\frac{1}{4}\times 2=\frac{6}{16},$$

$$P(2)=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$$

$\therefore$ Probability distribution is

x	0	1	2
$P(x)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

$$\text{Mean}=\Sigma xP(x)=0\times\frac{9}{16}+1\times\frac{6}{16}+2\times\frac{1}{16}$$

$$=0+\frac{6}{16}+\frac{2}{16}=\frac{8}{16}=\frac{1}{2}$$

$\therefore$ Mean $=\frac{1}{2}$

Value :

1. No, it may be good once or twice but not forever.
2. Honesty pays in a long run.

10. L.H.S. $=2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right)$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{\cos^2\frac{\alpha}{2}\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{\frac{1}{2}\sin\alpha\sin\left(\frac{\pi}{2}-\beta\right)}{\left(\frac{1+\cos\alpha}{2}\right)\left(\frac{1+\sin\beta}{2}\right)-\left(\frac{1-\cos\alpha}{2}\right)\left(\frac{1-\sin\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left(\frac{2\sin\alpha\cos\beta}{(1+\cos\alpha)(1+\sin\beta)-(1-\cos\alpha)(1-\sin\beta)}\right)$$

$$=\tan^{-1}\left(\frac{2\sin\alpha\cos\beta}{2\cos\alpha+2\sin\beta}\right)=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}\right)$$

= R.H.S.

11. $$|A|=\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}$$

$=1(1+\sin^2\theta)-\sin\theta(-\sin\theta+\sin\theta)+1(1+\sin^2\theta)$

$=2(1+\sin^2\theta)$

The value of $\sin^2\theta$ lies in the range of 0 and 1.

$\Rightarrow\ 0\le\sin^2\theta\le 1 \ \Rightarrow\ 0+1\le(1+\sin^2\theta)\le 1+1$

$\Rightarrow\ 1\le(1+\sin^2\theta)\le 2 \Rightarrow\ 2(1)\le 2(1+\sin^2\theta)\le 2(2)$

$\Rightarrow\ 2\le 2(1+\sin^2\theta)\le 4\ \Rightarrow\ 2\le|A|\le 4$

$\Rightarrow\ |A|\in[2,4]$

So, the value of $|A|$ lies in the interval [2, 4]

12. Let $I=\int_0^{\pi} x\sin^3 x\,dx$ ...(1)

Then, $I=\int_0^{\pi}(\pi-x)\sin^3(\pi-x)\,dx$

or $\quad I=\int_0^{\pi}(\pi-x)\sin^3 x\,dx$ ...(2)

Adding (1) and (2), we get

$$2I=\int_0^{\pi}\pi\sin^3 x\,dx=\pi\int_0^{\pi}\sin^2 x\cdot\sin x\,dx$$

$$\Rightarrow\quad 2I=\pi\int_0^{\pi}(1-\cos^2 x)\sin x-dx$$

Let $\cos x = t \Rightarrow \sin x dx = -dt$

$x = 0 \Rightarrow t = 1$ and $x = \pi \Rightarrow t = -1$

$$\Rightarrow\quad 2I=-\pi\int_1^{-1}(1-t^2)\,dt$$

$$\Rightarrow\quad 2I=\pi\int_{-1}^{1}(1-t^2)\,dt=\pi\left[t-\frac{t^3}{3}\right]_{-1}^{1}=\frac{4\pi}{3}.$$

Hence, $I=\frac{2\pi}{3}$

OR

Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)dx$

$= \int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)dx$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$x = 0 \Rightarrow t = 0,\ x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore\ I = \int_0^1 2t \tan^{-1} t\, dt = 2\int_0^1 t \tan^{-1} t\, dt$

Integrating by parts, we have

$$I = 2\left[\frac{t^2}{2}\tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{t^2}{2(1+t^2)}dt$$

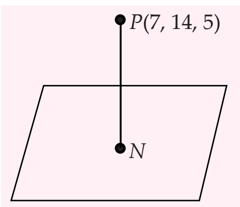
$$= 2\left[\frac{1}{2}\tan^{-1} 1 - 0\right] - \int_0^1 \frac{t^2}{(1+t^2)}dt$$

$$= 2\left[\frac{1}{2}\times\frac{\pi}{4}\right] - \int_0^1 \frac{t^2+1-1}{(1+t^2)}dt$$

$$= \frac{\pi}{4} - \int_0^1\left(1 - \frac{1}{(1+t^2)}\right)dt$$

$$= \frac{\pi}{4} - [t]_0^1 + [\tan^{-1} t]_0^1 = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

13. Any line through $P(7, 14, 5)$ and perpendicular to the plane $2x + 4y - z = 2$ is given by



$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda \text{(say)}.$$

Any point on this line is $N(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$.

If N is the foot of the perpendicular from P to the given plane, then it must lie on the plane $2x + 4y - z = 2$.

$\therefore\ 2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2 \Rightarrow \lambda = -3.$

Thus, we get the point $N(1, 2, 8)$ on the plane.

Hence, the foot of the perpendicular from $P(7, 14, 5)$ to the given plane is $N(1, 2, 8)$.

$\therefore$ Length of the perpendicular from P to the given plane $= PN = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$

$= \sqrt{189} = 3\sqrt{21}$ units.

14. Given $A = \{1, 2, 3, 4, ..., 9\} \subset N$, the set of natural numbers.

To show that R is an equivalence relation

Reflexivity :

Let (a, b) be an arbitrary element of $A \times A$. Then, we have: $(a, b) \in A \times A$

$\Rightarrow\ a, b \in A$

$\Rightarrow\ a + b = b + a$ (by commutativity of addition on $A \subset N$)

$\Rightarrow\ (a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in A \times A$

So, R is reflexive.

Symmetry :

Let $(a, b), (c, d) \in A \times A$ such that $(a, b)\ R\ (c, d)$.

i.e., $a + d = b + c$

$\Rightarrow\ b + c = a + d$

$\Rightarrow\ c + b = d + a$ (by commutativity of addition on $A \subset N$)

$\Rightarrow\ (c, d)\ R\ (a, b)$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in A \times A$. So, R is symmetric.

Transitivity :

Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

Now, $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ ...(1)

and $(c, d)\ R\ (e, f) \Rightarrow\ c + f = d + e$ ...(2)

Adding (1) and (2), we get

$(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow\ a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow\ (a, b)\ R\ (e, f)\ \forall\ (a, b), (c, d), (e, f) \in A \times A$

So, R is transitive on $A \times A$.

$\therefore\ R$ is an equivalence relation.

OR

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

From the table, the second row and second column are the same as the original set.

0*0 = 0, 1*0 = 0*1=1, 2*0 = 0*2 = 2, 3*0 = 0*3 = 3, 4*0 = 0*4 = 4, 0*5 = 5*0 = 5. Hence, '0' is the identity element of the operation '*'.

Now, the element '0' appears in the cell 1*5 = 5*1 = 0, 2*4 = 4*2 = 0, 3*3 = 0 and 0*0 = 0

Inverse element of 0 is 0, inverse element of 1 is 5, inverse element of 2 is 4, inverse element of 3 is 3, inverse element of 4 is 2, inverse element of 5 is 1.

15. Given, $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 5 & 3\\ 0 & 2 & 1\end{bmatrix}$

For finding the inverse, write $A = IA$

$$\therefore \quad \begin{bmatrix}1 & -1 & 0\\ 2 & 5 & 3\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 7 & 3\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to R_2 - 3R_3$, we get

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 2R_2$, we get

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 1 & -3\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix}A$$

$$\therefore \quad A^{-1}=\begin{bmatrix}-1 & 1 & -3\\ -2 & 1 & -3\\ 4 & -2 & 7\end{bmatrix} [\because\ A^{-1}A=I]$$

16. We have, $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{a}\cdot(\vec{b}-\vec{c})=0$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{b}-\vec{c}=\vec{0}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow \quad \vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$ ...(1)

Again, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=\vec{0}$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{a}\times(\vec{b}-\vec{c})=\vec{0}$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad (\vec{b}-\vec{c})=\vec{0}$ or $\vec{a}\,||\,(\vec{b}-\vec{c})$

$\Rightarrow \quad \vec{b}=\vec{c}$ or $\vec{a}\,||\,(\vec{b}-\vec{c})$ ...(2)

From (1) and (2), we get $\vec{b}=\vec{c}$

[$\because\ \ \vec{a}\perp(\vec{b}-\vec{c})$ and $\vec{a}\,||\,(\vec{b}-\vec{c})$ both cannot hold simultaneously].

17. Let $I=\int\frac{3x+1}{\sqrt{5-2x-x^2}}dx$ ...(1)

This integral is of the form $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

whose solution is

$$px+q=A\frac{d}{dx}(ax^2+bx+c)+B$$

Now, we write given integrand as

$$3x+1=A\frac{d}{dx}(5-2x-x^2)+B$$

$\Rightarrow \quad 3x + 1 = A(- 2 - 2x) + B$

$\Rightarrow \quad 3x + 1 = - 2Ax + (-2A + B)$

On equating the coefficients of x and constant term both sides, we get

$$3=-2A \Rightarrow A=-\frac{3}{2}$$

and $1=-2A+B \Rightarrow 1=-2\left(-\frac{3}{2}\right)+B \Rightarrow B=-2$

$\therefore$ Given integral can be rewritten as

$$I=\int\frac{-\frac{3}{2}(-2-2x)}{\sqrt{5-2x-x^2}}dx+\int\frac{-2}{\sqrt{5-2x-x^2}}dx$$

$$=-\frac{3}{2}\left[\frac{(5-2x-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]-\int\frac{2}{\sqrt{(\sqrt{6})^2-(x+1)^2}}dx$$

$$=-\frac{3}{2}\frac{(\sqrt{5-2x-x^2})}{1/2}-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$$

$$=-3(\sqrt{5-2x-x^2})-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$$

18. Here, $f(0) = c$

L.H.L. at $x = 0$

$$\underset{x\to 0^-}{\text{Lt}}\ f(x)=\underset{x\to 0}{\text{Lt}}\frac{\sin(a+1)x+\sin x}{x}$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\left\{\frac{\sin(a+1)x}{(a+1)x}\right\}\cdot(a+1)+\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}$$

$= (a + 1) + 1$ $\quad (\because \text{ If } x \to 0^- \Rightarrow (a+1)x \to 0^-)$

$= a + 1 + 1 = a + 2$

R.H.L. at $x = 0$

$$\underset{x\to 0^+}{\text{Lt}} f(x) = \underset{x\to 0}{\text{Lt}}\frac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}} = \underset{x\to 0}{\text{Lt}}\frac{\sqrt{x}\{\sqrt{1+bx}-1\}}{bx\sqrt{x}}$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{\sqrt{1+bx}-1}{bx}\right)\times\left(\frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\frac{1+bx-1}{bx(\sqrt{1+bx}+1)}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{1}{\sqrt{1+bx}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

Now, f is continuous at $x = 0$ if

$$\underset{x\to 0^-}{\text{Lt}} f(x) = f(0) = \underset{x\to 0^+}{\text{Lt}} f(x)$$

i.e., if $a + 2 = c = \frac{1}{2} \Rightarrow a = -\frac{3}{2}$ and $c = \frac{1}{2}$

Hence, for f to be continuous at $x = 0$, we must have

$a = -\frac{3}{2}$, $c = \frac{1}{2}$; b may have any real value.

OR

We have, $f(x) = x^3 + bx^2 + ax + 5$

On differentiating both sides w.r.t. x, we get

$f'(x) = 3x^2 + 2bx + a$

Since, the function satisfies the Rolle's theorem

$\therefore \quad f(1) = f(3)$

$\Rightarrow \quad 1^3 + b(1)^2 + a(1) + 5 = (3)^3 + b(3)^2 + a(3) + 5$

$\Rightarrow \quad b + a = 26 + 9b + 3a \Rightarrow 2a + 8b + 26 = 0$

$\Rightarrow \quad a + 4b = -13$ ...(1)

and $f'(c) = 0$

$$\Rightarrow f'\left(2+\frac{1}{\sqrt{3}}\right)=0$$

$$\Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right)+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$\Rightarrow 13+\frac{12}{\sqrt{3}}+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$\Rightarrow 2b\left(2+\frac{1}{\sqrt{3}}\right)+a=-13-\frac{12}{\sqrt{3}} \quad \text{... (2)}$$

On solving, (1) and (2), we get

$a = 11, b = -6$

19. We have, $\sec^2 y\frac{dy}{dx}+2x\tan y = x^3$

Let $\tan y = t \Rightarrow \sec^2 y\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx}+2tx = x^3$$

This is a linear differential equation of the form,

$\frac{dt}{dx}+Pt = Q,$

where $P = 2x$ and $Q = x^3$

Therefore, integrating factor

$\text{IF} = e^{\int Pdx} = e^{\int 2xdx} = e^{x^2}$

Solution of the differential equation is given by

$$t\,e^{x^2} = \int x^3e^{x^2}dx + C \quad \text{...(1)}$$

To solve $\int x^3e^{x^2}dx$

Let $x^2 = z \quad \Rightarrow \quad 2x\,dx = dz$

$$\Rightarrow \int x^3e^{x^2}dx = \frac{1}{2}\int ze^z dz$$

$$= \frac{1}{2}\left[ze^z - \int e^z dz\right]+C$$

$$= \frac{1}{2}\left[ze^z - e^z\right]+C$$

$$= \frac{1}{2}(x^2-1)e^{x^2}+C \quad \text{...(2)}$$

Thus, we have,

$$t\,e^{x^2} = \frac{1}{2}(x^2-1)e^{x^2}+C \quad \text{[from (1) and (2)]}$$

$$\Rightarrow t = \frac{1}{2}(x^2-1)+Ce^{-x^2}$$

$$\Rightarrow \tan y = \frac{1}{2}(x^2-1)+Ce^{-x^2} \quad [\because t = \tan y]$$

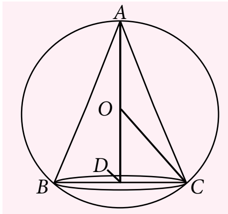
20. R is the radius of the given sphere with centre O and let V be the volume of the inscribed cone, h be its height and r be the radius of the base.

In the given figure, we have $OD = AD - AO = (h - R)$.

In ΔODC,

$\therefore \quad R^2 = (h - R)^2 + r^2$ or $r^2 = h(2R - h)$ ...(1)

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2(2R-h)$ [using (1)].



$\therefore \quad \dfrac{dV}{dh} = \dfrac{1}{3}\pi h(4R-3h)$, and $\dfrac{d^2V}{dh^2} = \left(\dfrac{4}{3}\pi R - 2\pi h\right)$.

For maxima or minima, we have $\dfrac{dV}{dh} = 0$.

Now, $\dfrac{dV}{dh} = 0 \Rightarrow \dfrac{1}{3}\pi h(4R-3h) = 0$

$\Rightarrow \quad h = 0$ or $(4R - 3h) = 0 \Rightarrow h = \dfrac{4}{3}R \quad [\because h \neq 0]$

And, $\left[\dfrac{d^2V}{dh^2}\right]_{h=(4/3)R} = -\dfrac{4\pi R}{3} < 0.$

So, V is maximum when $h = \dfrac{4}{3}R$ *i.e.*, when $3h = 2(2R)$

i.e., 3 times the height = 2 times the diameter of the sphere

Now, volume of the largest cone

$= \dfrac{1}{3}\pi \times \dfrac{16R^2}{9} \times \left(2R - \dfrac{4R}{3}\right) = \dfrac{32\pi R^3}{81}$

21. Let E = event of drawing a heart.

Then, $P(E) = \dfrac{13}{52} = \dfrac{1}{4}$ and $P(\bar{E}) = \left(1 - \dfrac{1}{4}\right) = \dfrac{3}{4}$

$\therefore \quad X$ denotes the number of hearts in three cards drawn.

Then, $X = 0, 1, 2$ or 3

$X = x_i$	p_i	
0	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}$	$=\frac{27}{64}$
1	$\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}+\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}+\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}$	$=\frac{27}{64}$
2	$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}+\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}$	$=\frac{9}{64}$
3	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}$	$=\frac{1}{64}$

$\therefore \quad$ Mean $(\mu) = \Sigma x_i p_i$

$= \left(0 \times \dfrac{27}{64}\right) + \left(1 \times \dfrac{27}{64}\right) + \left(2 \times \dfrac{9}{64}\right) + \left(3 \times \dfrac{1}{64}\right) = \dfrac{3}{4}$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$= \left(0 \times \dfrac{27}{64}\right) + \left(1^2 \times \dfrac{27}{64}\right) + \left(2^2 \times \dfrac{9}{64}\right) + \left(3^2 \times \dfrac{1}{64}\right) - \dfrac{9}{16} = \dfrac{9}{16}$

OR

Let A, E_1 and E_2 respectively denotes the event that the person suffers a heart attack, the selected person followed the course of yoga and meditation and the person adopted the drug prescription

$P(A) = \dfrac{40}{100} = 0.40$

$P(E_1) = P(E_2) = \dfrac{1}{2}$

$P(A/E_1) = 0.40 \times 0.70 = 0.28$

$P(A/E_2) = 0.40 \times 0.75 = 0.30$

Probability that the patient suffering heartattack followed course of meditation and yoga is

$P(E_1 | A) = \dfrac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$

$= \dfrac{\frac{1}{2}\times 0.28}{\frac{1}{2}\times 0.28 + \frac{1}{2}\times 0.30} = \dfrac{0.14}{0.14+0.15} = \dfrac{14}{29}$

Now, $P(E_2 | A) = \dfrac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$

$= \dfrac{\frac{1}{2}\times 0.30}{\frac{1}{2}\times 0.28 + \frac{1}{2}\times 0.30} = \dfrac{0.15}{0.14+0.15} = \dfrac{15}{29}$

Since $P(E_1|A) < P(E_2|A)$, the course of yoga and meditation is more beneficial for a person having heart attack.

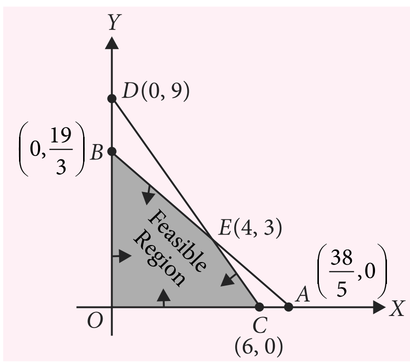
22. (i) If x machines of type A and y machines of type B are bought, then our problem is to maximize $Z = 50x + 40y$ subject to constraints

$x \geq 0, y \geq 0$

$1000x + 1200y \leq 7600 \Rightarrow 5x + 6y \leq 38$

$12x + 8y \leq 72 \Rightarrow 3x + 2y \leq 18$

Feasible region is shown shaded in the figure.



Rvp

Points	Value of $Z = 50x + 40y$
$O(0, 0)$	$50 \times 0 + 40 \times 0 = 0$
$B\left(0, \frac{19}{3}\right)$	$50 \times 0 + 40 \times \frac{19}{3} = 253.3$
$E(4, 3)$	$50 \times 4 + 40 \times 3 = 320$
$C(6, 0)$	$50 \times 6 + 40 \times 0 = 300$

i.e., when 4 machines of type A and 3 machines of type B are bought, the daily output would be maximum

(ii) (a) High productivity

(b) Time management

(c) Efficiency

23. We know that the lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

are coplanar $\Leftrightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0,$

and the equation of the plane containing these lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

Here, $x_1=0, y_1=2, z_1=-3; x_2=2, y_2=6, z_2=3; a_1=1,$ $b_1 = 2, c_1 = 3; a_2 = 2, b_2 = 3, c_2 = 4.$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}=0.$$

Hence, the given two lines are coplanar.

The equation of the plane containing these lines is

$$\begin{vmatrix} x-0 & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}=0 \Leftrightarrow \begin{vmatrix} x & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}=0$$

$\Rightarrow \quad x(8 - 9) - (y - 2)(4 - 6) + (z + 3)(3 - 4) = 0$

$\Rightarrow \quad -x + 2(y - 2) - (z + 3) = 0 \Rightarrow x - 2y + z + 7 = 0$

24. Let $y = f(x) = 4x^2 + 12x + 15, x \in N$

Now, firstly find the range of y.

Consider $y = (2x)^2 + 2 \times 2x \times 3 + 9 + 6$

$\Rightarrow \quad y = (2x + 3)^2 + 6 \Rightarrow (2x + 3)^2 = y - 6$

$\Rightarrow \quad 2x+3=\sqrt{y-6}$ [taking positive square root]

$\Rightarrow \quad x=\dfrac{\sqrt{y-6}-3}{2}$ [$\because \sqrt{y-6}$ is defined for $y > 6$]

Let $g : S \to N$ be defined by

$$g(y)=\frac{(\sqrt{y-6})-3}{2} \quad \text{...(1)}$$

Now, $gof(x) = g(4x^2 + 12x + 15) = g[(2x + 3)^2 + 6]$

$$=\frac{\sqrt{(2x+3)^2+6}-6-3}{2} \quad \text{[from (1)]}$$

$\Rightarrow \quad gof(x) = I_N$

and $\quad fog(y)=f\left(\dfrac{\sqrt{y-6}-3}{2}\right)$

$$=\left[\frac{2\{\sqrt{(y-6)}-3\}}{2}+3\right]^2+6=(\sqrt{y-6})^2+6=y$$

$\Rightarrow \quad fog(y) = I_S$

Hence, f is invertible and inverse of f is

$$f^{-1}(x)=\frac{(\sqrt{x-6})-3}{2}$$

25. Let $I=\displaystyle\int\frac{x^4dx}{(x-1)(x^2+1)}$

Consider the expression $\dfrac{x^4}{(x-1)(x^2+1)}$

$$\frac{x^4}{(x-1)(x^2+1)}=(x+1)+\frac{1}{(x-1)(x^2+1)} \quad \text{... (1)}$$

$$\frac{1}{(x-1)(x^2+1)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+1)} \quad \text{... (2)}$$

$1 = A(x^2 + 1) + (Bx + C)(x - 1)$

$= (A + B)x^2 + (C - B)x + A - C$

$A + B = 0, C - B = 0$ and $A - C = 1,$

$A=\frac{1}{2}, B=C=-\frac{1}{2}$

Substituting the values of A, B and C in (2), we have,

$$\frac{1}{(x-1)(x^2+1)}=\frac{1}{2(x-1)}-\frac{1}{2}\frac{x}{(x^2+1)}-\frac{1}{2(x^2+1)} \quad \text{...(3)}$$

$\Rightarrow \quad \dfrac{x^4}{(x-1)(x^2+1)}$

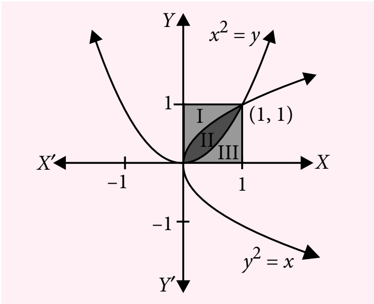
$$=(x+1)+\frac{1}{2(x-1)}-\frac{1}{2}\frac{x}{(x^2+1)}-\frac{1}{2(x^2+1)}$$

[From (1) and (3)]

$$\int\frac{x^4}{(x-1)(x^2+1)}dx=\int(x+1)dx+\int\frac{1}{2(x-1)}dx-\int\frac{x}{2(x^2+1)}dx-\int\frac{1}{2(x^2+1)}dx$$

$$=\frac{x^2}{2}+x+\frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C$$

OR



The points where the two parabolas meet in the first quadrant are obtained by solving the two equations $y = x^2$ ... (1) and $x = y^2$ ...(2)

From (1) and (2), we get

$x = (x^2)^2$

$\Rightarrow \quad x = x^4 \Rightarrow x^4 - x = 0$, *i.e.* $x(x^3 - 1) = 0 \Rightarrow x = 0, 1$

So, $y = 0, 1$

$\therefore$ The points where the two parabolas meet in the first quadrant are (0,0) and (1,1). The area gets divided into 3 parts as shown in the figure.

$$\text{Area I} = \int_0^1 (1-\sqrt{x})dx = \left[x - \frac{x^{3/2}}{3/2}\right]_0^1$$

$$= \left(1 - \frac{2}{3}\right) = \frac{1}{3}\text{sq. units}$$

$$\text{Area II} = \int_0^1 (\sqrt{x} - x^2)dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3}\text{sq. units}$$

$$\text{Area III} = \int_0^1 (x^2)dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\text{ sq. units}$$

Area I = Area II = Area III

$\therefore$ The curves $y = x^2$ and $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 1$ into three parts which are equal in area.

26. Given differential equation is

$$xe^{(y/x)} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0 \qquad ...(1)$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (1), we get

$$xe^{\frac{vx}{x}} - vx\sin\left(\frac{vx}{x}\right) + x\left(v + x\frac{dv}{dx}\right)\sin\left(\frac{vx}{x}\right) = 0$$

$$\Rightarrow xe^v - vx\sin(v) + vx\sin v + x^2\sin v\frac{dv}{dx} = 0$$

$$\Rightarrow x^2\sin v\frac{dv}{dx} = -xe^v \Rightarrow e^{-v}\sin v\,dv = -\frac{1}{x}dx$$

On integrating both sides, we get

$$\int e^{-v}\sin v\,dv = \int -\frac{1}{x}dx \qquad ... (2)$$

Let $I_1 = \int \underset{\text{II}}{e^{-v}} \cdot \underset{\text{I}}{\sin v}\,dv$

$$I_1 = \sin v\int e^{-v}dv - \int\left[\frac{d}{dv}(\sin v)\int e^{-v}dv\right]dv$$

[using integration by parts]

$$\Rightarrow I_1 = -e^{-v}\sin v + \int \underset{\text{II}}{e^{-v}}\,\underset{\text{I}}{\cos v}\,dv$$

$$= -e^{-v}\sin v + \cos v\int e^{-v}dv - \int\left[\frac{d}{dv}\cos v\int e^{-v}dv\right]dv$$

[using integration by parts]

$$\Rightarrow I_1 = -e^{-v}\sin v - e^{-v}\cos v - \int e^{-v}\sin v\,dv$$

$$\Rightarrow I_1 = -e^{-v}\sin v - e^{-v}\cos v - I_1$$

$$\Rightarrow I_1 = \frac{-e^{-v}}{2}(\sin v + \cos v) \qquad ... (3)$$

From equation (2) and (3), we get

$$-\frac{e^{-v}}{2}(\sin v + \cos v) = -\log|x| + C_1$$

$$\Rightarrow -\frac{e^{-y/x}}{2}\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] = -\log|x| + C_1$$

$$\left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow e^{-y/x}\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] = 2\log x - 2C_1$$

[multiplying both sides by –2]

$$\Rightarrow e^{-y/x}\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] = \log x^2 + C$$

[put $C = -2C_1$]

Also, given that $y = 0$ and $x = 1$

$\therefore \quad e^{-0}(\sin(0) + \cos(0)] = \log 1^2 + C$

$\Rightarrow \quad 1[0 + 1] = 0 + C \quad \Rightarrow \quad C = 1$

$\therefore \quad$ Required particular solution is

$$e^{-y/x}\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] = \log x^2 + 1$$

JEE WORK OUTS

1. $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$, then $\frac{dy}{dx} =$

(a) $\frac{2y-1}{\cos x}$ (b) $\frac{\cos x}{2y-1}$

(c) $\frac{2x-1}{\cos y}$ (d) $\frac{\cos y}{2y-1}$

2. The equation of the hyperbola referred to its axis as axis of co–ordinate and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(a) $x^2 - y^2 = 16$ (b) $x^2 - y^2 = 32$

(c) $x^2 - 2y^2 = 16$ (d) $x^2 - y^2 = -16$

3. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1 + x)^n$, then $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} =$

(a) $\frac{a_2}{a_2+a_3}$ (b) $\frac{a_2}{2(a_2+a_3)}$

(c) $\frac{2a_2}{a_2+a_3}$ (d) $\frac{2a_3}{a_2+a_3}$

4. $\frac{d}{dx}[\tan^{-1}\log_{10} x] =$

(a) $\frac{1}{[1+(\log_{10} x)^2]\cdot x\log_e 10}$ (b) $\frac{1}{1+\log_{10} x^2}$

(c) $\frac{1}{[1+\log_{10} x^2]\cdot x\log_{10} e}$ (d) None of these

5. If e, e' be the eccentricities of two conics $S = 0$ and $S' = 0$ and if $e^2 + e'^2 = 3$ then both S and S' are

(a) hyperbolas (b) ellipses

(c) parabolas (d) None of these

6. If $f(x) = \begin{cases} \int_0^x 1+|1-t|dt, & x>2 \\ 5x-7 & , x \le 2 \end{cases}$, then

(a) $f(x)$ is not continuous at $x = 2$

(b) f is differentiable everywhere

(c) RHL at $x = 2$ doesn't exist

(d) f is continuous but not differentiable at $x = 2$

7. The length of normal at θ on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ is

(a) $|a\sin^2\theta|$ (b) $|a\sin^2\theta \tan\theta|$

(c) $|a\sin^2\theta \cos\theta|$ (d) $|a\sin^3\theta \tan\theta|$

8. If A, B, C are the angles which a directed line makes with the positive directions of the co–ordinate axes, then $\sin^2 A + \sin^2 B + \sin^2 C =$

(a) 1 (b) 2

(c) 3 (d) None of these

9. Sum of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ to 16 terms is

(a) 346 (b) 446

(c) 546 (d) None of these

10. If $f(x) = \begin{cases} x & : x<0 \\ 1 & : x=0 \\ x^2 & : x>0 \end{cases}$, then $\lim_{x\to 0} f(x) =$

(a) 0 (b) 1

(c) doesn't exist (d) 2

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11. The tangent at any point on the curve $x^4 + y^4 = a^4$ cuts off intercepts p and q on the co–ordinate axes, then the value of $p^{-4/3} + q^{-4/3}$ is equal to

(a) $a^{-4/3}$ (b) $a^{-1/2}$

(c) $a^{1/2}$ (d) None of these

12. $\int_0^1 \frac{dx}{(x^2+1)^{3/2}} =$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) 1 (d) $\frac{-1}{\sqrt{2}}$

13. The sum of the series

$\frac{1}{3\times7} + \frac{1}{7\times11} + \frac{1}{11\times15} + \ldots\infty$ is

(a) 113 (b) 1/6

(c) 1/9 (d) 1/12

14. If $[x]$ denotes the greatest integer less than or equal to x, then the value of $\lim_{x\to0}(1-x+[x-1]+[1-x])$ is

(a) 0 (b) 1

(c) doesn't exist (d) –1

15. $\int x^n \log x\, dx =$

(a) $\frac{x^{n+1}}{n}[n\log x - 1] + C$

(b) $\frac{x^{n+1}}{n+1}[(n+1)\log x - 1] + C$

(c) $\frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1] + C$

(d) None of these

16. The orthogonal trajectories of the family of curves $a^{n-1} y = x^n$ are given by

(a) $x^n + n^2y$ = constant (b) $ny^2 + x^2$ = constant

(c) $n^2x + y^n$ = constant (d) $n^2x - y^n$ = constant

17. If each observation of a raw data, whose variance is σ^2, is multiplied by λ, then the variance of the new set is

(a) σ^2 (b) $\lambda^2\sigma^2$

(c) $\lambda + \sigma^2$ (d) $\lambda^2 + \sigma^2$

18. If the line $y = x\sqrt{3}$ cuts the curve $x^3 + y^3 + 3yx + 5x^2 + 3y^2 + 4x + 5y - 1 = 0$ at the points A, B and C, then $OA \cdot OB \cdot OC$ (where O is (0, 0)) is

(a) $\frac{4}{13}(3\sqrt{3}-1)$ (b) $3\sqrt{3}+1$

(c) $\frac{1}{\sqrt{3}}(2+7\sqrt{3})$ (d) None of these

19. A candidate appearing for the competitive test finds the following information on seeing the question paper :

(i) The total time is 3 hours

(ii) The paper has two sections – section A and section B.

(iii) To answer atleast two questions from section A and atleast three question from section B.

(iv) A question of first section carries 10 marks and that of section B carries 15 marks

(v) Time to answer a question of section A is 15 minutes and that of section B is 25 minutes.

(vi) The maximum number of question to be answered is 10.

The maximum marks he can secure is

(a) 115 (b) 125

(c) 100 (d) None of these

20. Equation of a line which is parallel to the line common to the pair of lines given by $6x^2 - xy - 12y^2 = 0$ and $15x^2 + 14xy - 8y^2 = 0$ and the sum of whose intercepts on the axes is 7, is

(a) $2x - 3y = 42$ (b) $3x + 4y = 12$

(c) $5x - 2y = 10$ (d) None of these

21. The roots of $\begin{vmatrix} x & a & b & 1 \\ \lambda & x & b & 1 \\ \lambda & \mu & x & 1 \\ \lambda & \mu & v & 1 \end{vmatrix} = 0$ are independent of

(a) λ, μ, v (b) a, b

(c) λ, μ, v, a, b (d) None of these

22. The incentre of the triangle formed by the co-ordinate axes and $3x + 4y = 12$ is

(a) (1/2, 1/2) (b) (1, 1)

(c) (1, 1/2) (d) (1/2, 1)

23. $\int \frac{xe^x}{(x+1)^2}dx =$

(a) $e^x/(x + 1) + C$ (b) $e^x/(x + 1)^2 + C$

(c) $-e^x/(x + 1)^3 + C$ (d) None of these

24. The equation of the circle passing through (1, 0), (0, 1) and having smallest possible radius is

(a) $x^2 + y^2 - x - y = 0$ (b) $x^2 + y^2 + x + y = 0$

(c) $x^2 + y^2 - 2x - y = 0$ (d) $x^2 + y^2 - x - 2y = 0$

25. If α, β are roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, the value of $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$ is

(a) $p^2 - q^2$ (b) $q^2 - p^2$

(c) p^2 (d) q^2

26. A line meets the co–ordinate axes in A and B. A circle is circumscribed about the ΔAOB. The distance from the end points of the side AB to the line touching the circle at origin O are equal to p and q respectively. The diameter of the circle is

(a) $p(p + q)$ (b) $q(p + q)$

(c) $p + q$ (d) $\frac{p+q}{2}$

27. Area between the curves $y = x^3$ and $y = \sqrt{x}$ is

(a) 5/12 (b) 5/3

(c) 5/4 (d) None of these

28. The equation of the directrix of the parabola $x^2 - 4x - 3y + 10 = 0$ is

(a) $y = -5/4$ (b) $y = 5/4$

(c) $y = -3/4$ (d) $x = 5/4$

29. If $a\cos^2(3\alpha) + b\cos^4\alpha = 16\cos^6\alpha + 9\cos^2\alpha$ is an identity, then

(a) $a = 1, b = 24$ (b) $a = 3, b = 24$

(c) $a = 4, b = 2$ (d) $a = 7, b = 18$

30. Tangents are drawn to the circle $x^2 + y^2 = 25$ from the point (13, 0). They include an angle

(a) $\tan^{-1}(5/12)$ (b) $\tan^{-1}(12/5)$

(c) $2\tan^{-1}(5/12)$ (d) None of these

31. $1 + 4 + 13 + 40 + \ldots n$ terms is

(a) $\left(\frac{3^{n+1} - 3 - 2n}{4}\right)$ (b) $\left(\frac{3^{n+1} + 3 - 2^n}{4}\right)$

(c) $\left(\frac{3^{n+1} + 3 + 2^n}{4}\right)$ (d) $\left(\frac{3^{n-1} - 3 + 2^n}{4}\right)$

32. The eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre of the ellipse is 2, is

(a) $\pi/4$ (b) $3\pi/2$

(c) $5\pi/3$ (d) $7\pi/6$

33. Two teams are to play a series of 5 matches between them. A match ends in a win or loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain 'n' people, where 'n' is

(a) 81 (b) 243

(c) 486 (d) None of these

34. Odds 8 to 5 against a person who is 40 years old living till he is 70 and 4 to 3 against another person now 50 till he will be living 80. Probability that one of them will be alive next 30 years is

(a) 59/91 (b) 44/91

(c) 51/91 (d) 32/91

35. Solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is

(a) $\frac{y-x}{1+xy} = k$ (b) $xy = k$

(c) $x^2 - y^2 = k$ (d) None of these

36. The number of ways in which 6 different balls can be put in two boxes of different sizes so that no box remains empty is

(a) 62 (b) 64

(c) 36 (d) None of these

37. The number of even proper divisors of 1008 is

(a) 23 (b) 24

(c) 22 (d) None of these

38. The solution of the differential equation $2x + \frac{dy}{dx} - y = 3$; given $y(0) = -1$ represents

(a) straight line (b) circle

(c) parabola (d) ellipse

39. If the three consecutive coefficients in the expansion of $(1 + x)^n$ are 28, 56 and 70, then 'n' equals

(a) 6 (b) 4

(c) 8 (d) 10

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Class XI

CONCEPT BOOSTERS

CONIC SECTION

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This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CONIC SECTION

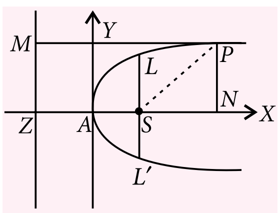
The locus of a point, which moves so that its distance from a fixed point is always in a constant ratio to its distance from a fixed straight line, not passing through the fixed point is called a conic section.

- The fixed point is called the focus.
- The fixed straight line is called the directrix.
- The constant ratio is called the eccentricity and is denoted by e.
- When the eccentricity is unity *i.e.*, $e = 1$, the conic is called a parabola ; when $e < 1$ the conic is called an ellipse and when $e > 1$, the conic is called a hyperbola.
- The straight line passing through the focus and perpendicular to the directrix is called the axis of the parabola.
- A point of intersection of a conic with its axis is called vertex.

PARABOLA

Standard Equation of a Parabola

Let S be the focus, ZM be the directrix and P be the moving point. Draw SZ perpendicular from S on the directrix. Then SZ is the axis of the parabola.



Now the middle point of SZ, say A, will lie on the locus of P, *i.e.*, $AS = AZ$. Take A as the origin, the x-axis along AS, and the y-axis along the perpendicular to AS at A, as in the figure.

Let $AS = a$, so that $ZA = a$. Let (x, y) be the coordinates of the moving point P.

Then $MP = ZN = ZA + AN = a + x$. But by definition

$$MP = PS \Rightarrow MP^2 = PS^2.$$

So that, $(a + x)^2 = (x - a)^2 + y^2$.

Hence, the equation of parabola is $y^2 = 4ax$.

Latus Rectum

In the given figure, LSL' is the latus rectum.

Also $LSL' = 2\sqrt{4a.a} = 4a$ = double ordinate through the focus S.

Note :

- Any chord of the parabola $y^2 = 4ax$ perpendicular to the axis of the parabola is called double ordinate.

Four common forms of a parabola

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex	$(0 , 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Eqn. of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Eqn. of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$

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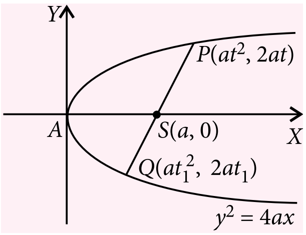
Parametric Coordinates

Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$ and we refer to it as the point 't'. Here, t is a parameter, *i.e.*, it varies from point to point.

Focal Chord

Any chord to $y^2 = 4ax$ which passes through the focus is called a focal chord of the parabola $y^2 = 4ax$.



Let $y^2 = 4ax$ be the equation of a parabola and $(at^2, 2at)$ a point P on it. Suppose the coordinates of the other extremity Q of the focal chord through P are $(at_1{}^2, 2at_1)$.

Then, PS and SQ, where S is the focus $(a, 0)$ have the same slopes.

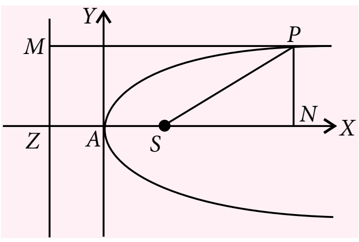
$$\Rightarrow \quad \frac{2at-0}{at^2-a}=\frac{2at_1-0}{at_1^2-a} \quad \Rightarrow \quad tt_1{}^2 - t = t_1t^2 - t_1$$

$\Rightarrow \quad (tt_1 + 1)(t_1 - t) = 0$

Hence $t_1 = -1/t$, *i.e.* the point Q is $(a/t^2, -2a/t)$.

The extremities of a focal chord of the parabola $y^2 = 4ax$ may be taken as the points t and $-1/t$.

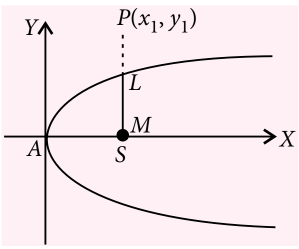
Focal Distance of Any Point : The focal distance of any point P on the parabola $y^2 = 4ax$ is the distance between the point P and the focus S, *i.e.* PS.



Thus the focal distance $= PS = PM = ZN = ZA + AN$
$= a + x$

POSITION OF A POINT RELATIVE TO THE PARABOLA

Consider the parabola $y^2 = 4ax$. If (x_1, y_1) is a given point and $y_1{}^2 - 4ax_1 = 0$, then the point lies on the parabola. But when $y_1^2 - 4ax_1 \neq 0$, we draw the ordinate PM meeting the curve at L. Then P will lie outside the parabola if $PM > LM$,



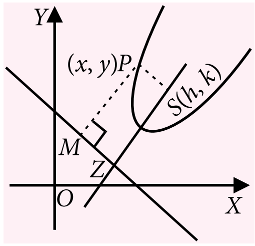
i.e., $PM^2 - LM^2 > 0$.

Now, $PM^2 = y_1{}^2$ and $LM^2 = 4ax_1$ by virtue of the parabola. Substituting these values in equation of parabola, the condition for P to lie outside the parabola becomes $y_1^2 - 4ax_1 > 0$.

Similarly, the condition for P to lie inside the parabola is $y_1^2 - 4ax_1 < 0$.

GENERAL EQUATION OF A PARABOLA

Let (h, k) be the focus S and $lx + my + n = 0$, the equation of the directrix ZM of a parabola. Let (x, y) be the coordinates of any point P on the parabola.



Then the relation, PS = distance of P from ZM, gives

$$(x - h)^2 + (y - k)^2 = (lx + my + n)^2/(l^2 + m^2)$$

$\Rightarrow \quad (mx - ly)^2 + 2gx + 2fy + d = 0$

This is the general equation of a parabola. It is clear that second-degree terms in the equation of a parabola form a perfect square. The converse is also true, *i.e.* if in an equation of the second degree, the second degree terms form a perfect square then the equation represents a parabola, unless it represents two parallel straight lines.

Note : The general equation of second degree *i.e.* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $D \neq 0$ and $h^2 = ab$, $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Special case

Let the vertex be (h, k) and the axis be parallel to the x-axis. Then the equation of parabola is given by $(y - k)^2 = 4a(x - h)$ which is equivalent to $x = Ay^2 + By + C$

If three points are given we can find A, B and C.

Similarly, when the axis is parallel to the y-axis, the equation of parabola is $y = A'x^2 + B'x + C'$.

TANGENT DRAWN AT A POINT LYING ON A GIVEN PARABOLA

- If $P(x_1, y_1)$ be a point on the parabola $y^2 = 4ax$, then the equation of the tangent at P is $yy_1 = 2a(x + x_1)$
- If $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then slope of the tangent at P

$$= \frac{2a}{2at} = \frac{1}{t} \qquad \left(\because \frac{dy}{dx} = \frac{2a}{y}\right)$$

and hence its equation is

$$y - 2at = \left(\frac{1}{t}\right)(x - at^2)$$

i.e. $yt = x + at^2$ .... (1)

If we substitute m for $1/t$, in equation (1), we have the following result. The equation

$$y = mx + \left(\frac{a}{m}\right) \quad .... (2)$$

- **Point of Intersection of Tangents at 't_1' & 't_2'**

  Equations of the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are

  $yt_1 = x + at_1{}^2$ .... (3)

  and $yt_2 = x + at_2^2$ .... (4)

  respectively.

  Solving equations (3) and (4) gives the coordinates of the intersection point of these two tangents as $(at_1t_2, a(t_1 + t_2))$

  Note:

  - If the tangents at t_1 & t_2 are at right angles then $t_1t_2 = -1$.
  - If the chord joining t_1, t_2 subtends a right angle at the vertex then $t_1t_2 = -4$

INTERSECTION OF A LINE AND A PARABOLA

Let the parabola be $y^2 = 4ax$ .... (1)

and the given line be $y = mx + c$ .... (2)

Eliminating y from (1) and (2), then

$(mx + c)^2 = 4ax$

or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ .... (3)

This equation is quadratic in x, gives two values of x which shows that every straight line will cut the parabola in two points may be real, coincident or imaginary according as discriminant of (3) $>, =, < 0$

i.e. $4(mc - 2a)^2 - 4m^2c^2 >, =, < 0$

or $4a^2 - 4amc >, =, < 0$

or $a >, =, < mc$ .... (4)

Note : If $m = 0$ then equation (3) gives

$$-4ax + c^2 = 0 \quad \text{or} \quad x = \frac{c^2}{4a}$$

which gives only one value of x and so every line parallel to x-axis cuts the parabola only in one real point.

- **Condition of Tangency**

  If the line (2) touches the parabola (1), then equation (3) has equal roots

  $\therefore$ Discriminant of (3) = 0

  $\Rightarrow 4(mc - 2a)^2 - 4m^2c^2 = 0$

  $\Rightarrow -4amc + 4a^2 = 0$

  $$\Rightarrow c = \frac{a}{m},\ m \neq 0 \quad ....(5)$$

  so, the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ (which is condition of tangency).

  Substituting the value of c from (5) in (2) then

  $$y = mx + \frac{a}{m},\ m \neq 0 \quad ....(6)$$

  Hence the line $y = mx + \frac{a}{m}$ will always be a tangent to the parabola $y^2 = 4ax$.

- **The point of contact**

  The point of contact of the tangents at 't' is $(at^2, 2at)$. In terms of slope 'm' of the tangent the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, $(m \neq 0)$.

EQUATION OF NORMAL TO THE PARABOLA

If $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then

$$\text{Slope of the tangent at } P = \frac{2a}{2at} = \frac{1}{t} \qquad \left[\because \frac{dy}{dx} = \frac{2a}{y}\right]$$

Therefore, slope of the normal at $P = -t$ and its equation is

$y - 2at = -t(x - at^2)$

i.e. $y = -tx + 2at + at^3$ ....(1)

If we substitute m for $-t$ in equation (1), we have the following result.

the equation

$y = mx - 2am - am^3$ ....(2)

is a normal. Real or imaginary can be drawn from any point to a given parabola and the algebraic sum of the ordinates of the feet of these three normals is zero.

If the normal (2) passes through the point (x_1, y_1),we have

$y_1 = mx_1 - 2am - am^3$

i.e. $am^3 + (2a - x_1)\, m + y_1 = 0$ ....(3)

The equation (2) gives three values of m, real or imaginary, If m_1, m_2 and m_3 be the roots of equation (3), then we have $m_1 + m_2 + m_3 = 0$.

Hence, the sum of the ordinates of the feet of these normals $= -2a(m_1 + m_2 + m_3) = 0$

Note:

- If normal at the point 't_1' meets the parabola again at 't_2' then $t_2 = -t_1 - \dfrac{2}{t_1}$
- If the normals at t_1 & t_2 meet again on the parabola then $t_1t_2 = 2$
- The point of intersection of the normals to the parabola $y^2 = 4ax$ at 't_1' and 't_2' is $[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$.

RULE FOR TRANSFORMING AN EQUATION FOR THE VARIOUS FORMS OF THE PARABOLA

If any equation derived for the parabola $y^2 = 4ax$, $(a > 0)$ is given by

$$E(x, y, a) = 0 \qquad ....(1)$$

then the same equation for the parabola $y^2 = -4ax$ will be

$$E(x, y, -a) = 0 \qquad ....(2)$$

For the parabola $x^2 = 4ay$ will be

$$E(y, x, a) = 0 \qquad ....(3)$$

and for the parabola $x^2 = -4ay$ will be

$$E(y, x, -a) = 0 \qquad ....(4)$$

If the coordinates of the vertex be (h, k), then substitute $(x - h)$ and $(y - k)$ for x and y respectively.

Using the above rule for the equation of general tangent to the parabola $y^2 = -4ax$, equation is

$$y = mx - \frac{a}{m}, \qquad ....(5)$$

and to the parabola $(y - k)^2 = 4a(x - h)$, equation is

$$(y - k) = m(x - h) + \frac{a}{m}, \qquad ....(6)$$

and to the parabola $x^2 = 4ay$, equation is

$$x = my + \frac{a}{m} \qquad ....(7)$$

Similarly, equation of a general normal to the parabola $y^2 = -4ax$ is

$$y = mx + 2am + am^3, \qquad ....(8)$$

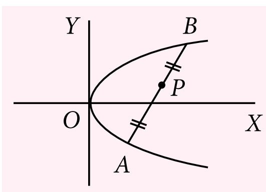
to the parabola $x^2 = 4ay$, equation is

$$x = my - 2am - am^3 \qquad ....(9)$$

and so on.

EQUATION OF THE CHORD WHOSE MID-POINT IS GIVEN

Equation of the chord of the parabola $y^2 = 4ax$ whose mid-point is (x_1, y_1) is given by

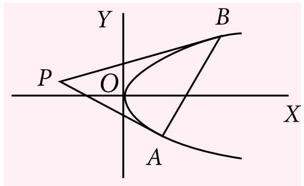


$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

or $T = S_1$

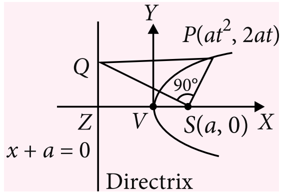
CHORD OF CONTACT

If PA & PB be the tangents through point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$, then the equation of the chord of contact AB is $yy_1 - 2a(x + x_1) = 0$ or $T = 0$

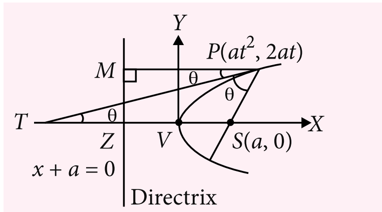


SOME STANDARD PROPERTIES OF THE PARABOLA

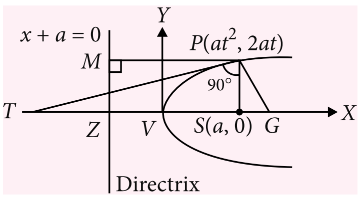
- The portion of a tangent to a parabola intercepted between the directrix and the curve subtends a right angle at the focus.



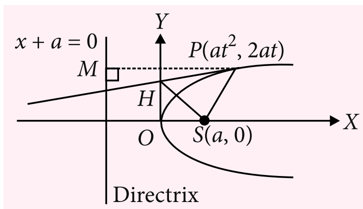
- The tangent at any point P of a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.



- The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.
- If S be the focus of the parabola and tangent and normal at any point P meet its axis in T and G respectively, then $ST = SG = SP$.



- If S be the focus and SH be perpendicular to the tangent at P, then H lies on the tangent at the vertex and $SH^2 = OS{\cdot}SP$, where O is the vertex of the parabola.



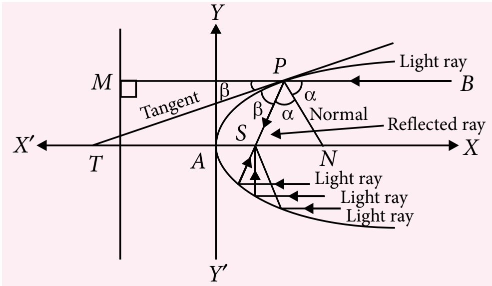
- The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.
- Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.
- If the tangents at P and Q meet at T, then
  1. TP and TQ subtend equal angles at the focus S.
  2. $ST^2 = SP \cdot SQ$
  3. The triangles SPT and STQ are similar.
- Tangents and Normal at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.
- Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is $2a = \dfrac{2bc}{b+c}$ i.e $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a}$.
- The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.
- The orthocenter of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix and has the coordinates $(-a, a(t_1+t_2+t_3+t_1t_2t_3))$.
- The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
- If normal drawn to a parabola passes through a point $P(h, k)$, then $k = mh - 2am - am^3$ i.e. $am^3 + m(2a-h) + k = 0$.

  Then it gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a}$; $m_1m_2m_3 = -\dfrac{k}{a}$, where m_1, m_2 and m_3 are the slopes of the three concurrent normal.

  Note that the algebraic sum of the
  1. slopes of three concurrent normal is zero.
  2. ordinates of the three co-normal points on the parabola is zero.
  3. centroid of the Δ formed by three co-normal points lies on the x-axis.

REFLECTION PROPERTY OF A PARABOLA

The tangent (PT) and normal (PN) of the parabola $y^2 = 4ax$ at P are the internal and external bisectors of $\angle SPM$ and BP is parallel to the axis of the parabola and $\angle BPN = \angle SPN$.



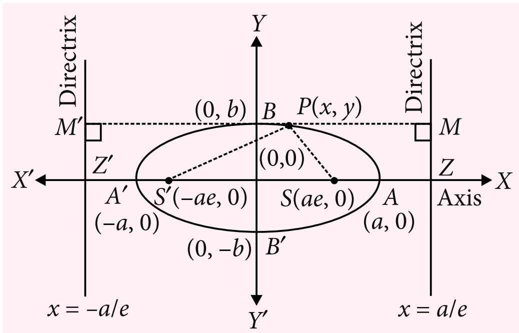
ELLIPSE

DEFINITION

An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point is in a constant ratio (less than one) to its distance from a fixed line. The fixed point is called the focus and fixed line is called the directrix and the constant ratio is called the eccentricity of the ellipse.

EQUATION OF AN ELLIPSE

Standard Equation of Ellipse



$\therefore \quad \dfrac{SP}{PM} = e$ or, $(SP)^2 = e^2(PM)^2$

$\Rightarrow \quad (x-ae)^2 + (y-0)^2 = e^2\left(\dfrac{a}{e}-x\right)^2$

$\Rightarrow \quad x^2(1-e^2) + y^2 = a^2(1-e^2)$

$\Rightarrow \quad \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)} = 1$ or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,$

where $b^2 = a^2(1-e^2)$

This is the standard equation of an ellipse. AA' and BB' are called the major and minor axes of the ellipse and their lengths will be given by $2a$ and $2b$ respectively. The minor axis and major axis taken together are called principal axes.

Here $b < a$ and $A(a, 0)$ and $A'(-a, 0)$ are the vertices of the ellipse.

Various terms related with an Ellipse

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, a > b$

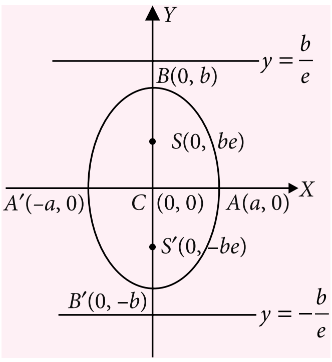
Rvp

- **Centre :** The point which bisects each chord of the ellipse is called the centre (0, 0).
- **Directrix :** ZM and $Z'M'$ are two directrices and their equations are $x = \frac{a}{e}$ and $x = \frac{-a}{e}$, respectively.
- **Focus :** $S(ae, 0)$ and $S'(-ae, 0)$ are two foci of the ellipse.
- **Length of Latus Rectum :** Length of latus rectum is given by $\frac{2b^2}{a}$.
- Relation between constant a, b and e
  $b^2 = a^2(1 - e^2)$
  $\therefore\ e = \frac{\sqrt{a^2 - b^2}}{a}$
- **Focal distances :** The focal distance of the point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $a + ex$ and $a - ex$.

Another form of Ellipse

The standard equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $a < b$

In this case, major axis is $BB' = 2b$ which is along y-axis and minor axis is $AA' = 2a$ which is along x-axis. Foci are $S(0, be)$ and $S'(0, -be)$ and directrices are $y = b/e$ and $y = -b/e$.



General equation of the ellipse

The general equation of an ellipse, whose focus is (h, k), the directrix is the line $ax + by + c = 0$ and the eccentricity e is given by

$$(x - h)^2 + (y - k)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}.$$

The Condition for a second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in x and y to represent an ellipse is given by

$$h^2 - ab < 0 \text{ and } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

Auxiliary Circle and Eccentric Angle :

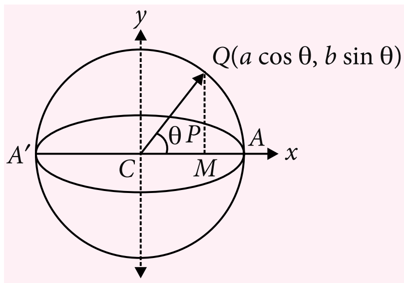
A circle described on major axis of an ellipse as diameter is called the auxiliary circle. Equation of the auxiliary circle is $x^2 + y^2 = a^2$. Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P and Q are called as the Corresponding Points on the ellipse and the auxiliary circle respectively and 'θ' is called the Eccentric Angle of the point P on the ellipse $(0 \le \theta < 2\pi)$.

Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$.

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

PARAMETRIC EQUATION OF AN ELLIPSE

Clearly $x = a\cos\theta$, $y = b\sin\theta$ satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all real values of θ. Moreover, any point on the ellipse can be represented as $(a\cos\theta, b\sin\theta)$, $0 \le \theta < 2\pi$. Hence the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x = a\cos\theta$, $y = b\sin\theta$, where θ is the parameter.



Equation of the chord of the ellipse whose eccentric angles are θ and φ is

$$\frac{x}{a}\cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

POINT AND ELLIPSE

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of an ellipse.

The point lies outside, on or inside the ellipse according

as $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0, < 0$ respectively.

ELLIPSE AND LINE

Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the given line be $y = mx + c$.

Solving the line and ellipse, we get

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

i.e., $(a^2m^2 + b^2)x^2 + 2mca^2 x + a^2(c^2 - b^2) = 0$

above equation being a quadratic in x.

Its discriminant $= 4m^2c^2a^4 - 4a^2(a^2m^2 + b^2)(c^2 - b^2)$

$= -4a^2b^2\{c^2 - (a^2m^2 + b^2)\} = 4a^2b^2\{(a^2m^2 + b^2) - c^2\}$

Hence the line intersects the ellipse in 2 distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

$\therefore \quad y = mx \pm \sqrt{(a^2m^2 + b^2)}$ touches the ellipse and condition for tangency is $c^2 = a^2m^2 + b^2$

Moreover the line $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ touches

the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

$$\left(\frac{\pm a^2 m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

Note:

- $x \cos\alpha + y \sin\alpha = p$ is a tangent if $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.
- $lx + my + n = 0$ is a tangent to the ellipse if $n^2 = a^2l^2 + b^2m^2$.

EQUATION OF THE TANGENT

- The equation of the tangent at any (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
- The equation of tangent at 'ϕ' is
  $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$
- Point of intersection of tangents to ellipse at 'θ' and 'ϕ' is $\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right)$

EQUATION OF THE NORMAL

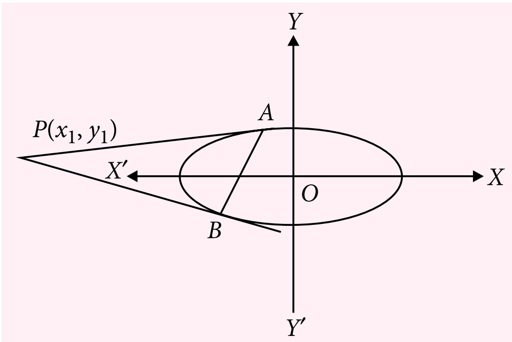
- The equation of the normal at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$
- The equation of the normal at 'ϕ' is $ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2$

EQUATION OF CHORD WITH MID POINT (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose mid point is (x_1, y_1) is given by $T = S_1$, where $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$ and $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$.

CHORD OF CONTACT

If PA and PB be the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1, y_1)

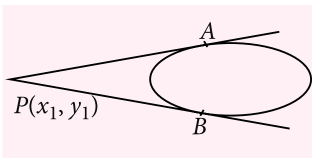


PAIR OF TANGENTS

Let $P(x_1, y_1)$ be any point outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let a pair of tangents PA, PB be drawn to it from P. Then, the equation of pair of tangents of PA and PB is given by

$$SS_1 = T^2, \text{ where } S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

$$S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ and } T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$



Note: The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are perpendicular to each

other is called the director circle and its equation is given by $x^2 + y^2 = a^2 + b^2$.

DIRECTOR CIRCLE

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ *i.e.* a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

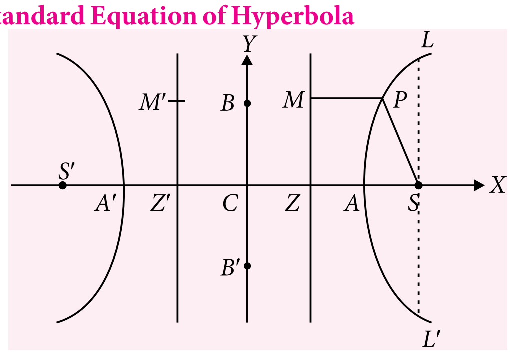
HYPERBOLA

DEFINITION

A hyperbola is the locus of a point which moves such that its distance from a fixed point called focus is always e times ($e > 1$) its distance from a fixed line is called directrix.

VARIOUS FORMS OF HYPERBOLA

Standard Equation of Hyperbola



$$\frac{SP}{PM} = e \text{ or } SP^2 = e^2 PM^2$$

$\Rightarrow \quad (x - ae)^2 + y^2 = e^2 (x - a/e)^2$

or $x^2(1 - e^2) + y^2 = a^2 (1 - e^2)$ *i.e.*,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 \qquad \text{....(i)}$$

Since $e > 1$, $e^2 - 1$ is positive. Let $a^2 (e^2 - 1) = b^2$. Then the equation (i) becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The eccentricity e of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by the relation $e^2 = \left(1 + \frac{b^2}{a^2}\right)$.

VARIOUS TERMS RELATED WITH HYPERBOLA

Foci: $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$

Directrices: ZM and $Z'M'$ are two directrices and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

Vertices: $A = (a, 0)$ and $A' = (-a, 0)$

Transverse Axis: The line segment $A'A$ is called transverse axis of the hyperbola. Length of transverse axis is $2a$.

Conjugate Axis: The line segment BB' is called the conjugate axis of the hyperbola. Length of conjugate axis is $2b$.

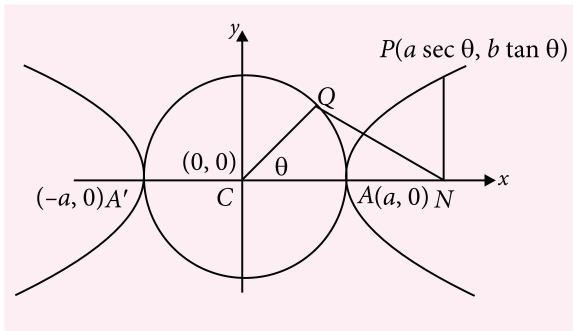
The transverse axis & the conjugate axis of the hyperbola together are called principal axes of the hyperbola.

Length of Latus Rectum :

Length of latus rectum $= \frac{2b^2}{a}$ and $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, \frac{-b^2}{a}\right)$ are end points of latus rectum.

Auxiliary Circle

A circle drawn with centre C and AA' as a diameter is called the Auxiliary Circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.



Note: From the figure, P and Q are called the "Corresponding Points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola .

Parametric Coordinates

The equations $x = a \sec \theta$ and $y = b \tan \theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. In other words, $(a \sec \theta, b \tan \theta)$ is a point on the hyperbola for all values of $\theta \neq (2n+1)\frac{\pi}{2}, n \in I$. The point $(a \sec \theta, b \tan \theta)$ is briefly called the point θ.

Note: Equation of a chord joining θ_1 & θ_2 is

$$\frac{x}{a} \cos\frac{\theta_1 - \theta_2}{2} - \frac{y}{b} \sin\frac{\theta_1 + \theta_2}{2} = \cos\frac{\theta_1 + \theta_2}{2}.$$

Note : Since the fundamental equation to the hyperbola only differs from that to the ellipse is having $-b^2$ instead of b^2 it will be found that many proposition for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

General Equation of Hyperbola

The equation of hyperbola, whose focus is point (h, k), directrix is $lx + my + n = 0$ and eccentricity 'e' is given by

$$(x - h)^2 + (y - k)^2 = \frac{e^2(lx+my+n)^2}{(l^2+m^2)}$$

CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

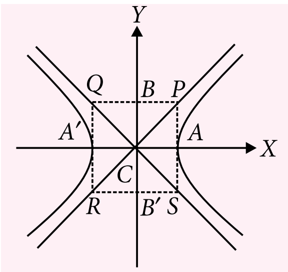
e.g., $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

are conjugate hyperbolas of each other.

Note: If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1.$

ASYMPTOTES

Definition: If the length of perpendicular drawn from a point on the hyperbola to a straight line tends to zero as the point moves to infinity. The straight line is called asymptotes.



$\therefore$ The equation of asymptotes are $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0\cdot$

Obviously angle between the asymptotes is $2\tan^{-1}(b/a)$.

If we draw lines through B, B' parallel to the transverse axis and through A, A' parallel to the conjugate axis, then $P(a, b)$, $Q(-a, b)$, $R(-a, -b)$ and $S(a, -b)$ all lie on the asymptotes $x^2/a^2 - y^2/b^2 = 0$ so asymptotes are diagonals of the rectangle $PQRS$. This rectangle is called associated rectangle.

Note: $H\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right), C\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\right)$ and

$A\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=0$

Clearly, $C + H = 2A$

{H = hyperbola, C = Conjugate hyperbola,
A = Asymptotes.}

Particular Case :

When $b = a$, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ is $y = \pm x$ which are at right angles.

- Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.
- If a hyperbola is equilateral, then the conjugate is also equilateral.
- A hyperbola and its conjugate have the same asymptote.
- The equation of the pair of asymptotes differ from the hyperbola and the conjugate hyperbola by the same constant only.
- The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.
- The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.
- Asymptotes are the tangent to the hyperbola from the centre.
- A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as,

  let $f(x, y) = 0$ represents a hyperbola. Find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$. Then the point of intersection of $\frac{\delta f}{\delta x}=0$ and $\frac{\delta f}{\delta y}=0$ gives the centre of the hyperbola.

RECTANGULAR OR EQUILATERAL HYPERBOLA

A hyperbola is called rectangular if its asymptotes are at right angles. The asymptotes of $x^2/a^2 - y^2/b^2 = 1$ are $y = \pm(b/a)x$, so they are perpendicular if $-b^2/a^2 = -1$ *i.e.*, $b^2 = a^2$, *i.e.*, $a = b$. Hence equation of a rectangular hyperbola can be written as $x^2 - y^2 = a^2$

Some important observations of rectangular hyperbola are as under :

- $a^2 = a^2(e^2 - 1)$ gives $e^2 = 2$ *i.e.*, $e = \sqrt{2}$.
- Asymptotes are $y = \pm x$.
- Rotating the axes by an angle $-\pi/4$ about the same origin, equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is reduced to $xy = a^2/2$ or $xy = c^2$, $(c^2 = a^2/2)$.
- In $xy = c^2$, asymptotes are coordinate axes.
- Rectangular hyperbola is also called equilateral hyperbola.
- Rectangular hyperbola referred to its asymptotes as axis of coordinates.
- Equation is $xy = c^2$ with parametric representation $x = ct, y = \frac{c}{t},\ t \in R \sim \{0\}$.

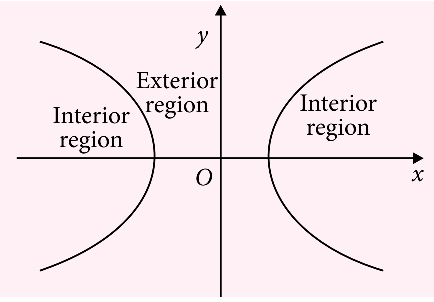
- Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$
- Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$ and at $P(t)$ is $\frac{x}{t}+ty=2c$.
- Chord with a given middle point as (h, k) is $kx + hy = 2hk$.
- Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.
- Vertex of this hyperbola is (c, c) and $(-c, -c)$; focus is $(\sqrt{2}c,\sqrt{2}c)$ and $(-\sqrt{2}c,-\sqrt{2}c)$, the directrices are $x+y=\pm\sqrt{2}c$ and $l(L.R.)=2\sqrt{2}c = T.A. = C.A.$

POSITION OF A POINT *P* W.R.T HYPERBOLA

Let $S = 0$ be the hyperbola and $P(x_1, y_1)$ be the point and $S_1 \equiv S(x_1, y_1)$.

Then

$S_1 < 0 \Rightarrow P$ is in the exterior region
$S_1 > 0 \Rightarrow P$ is in the interior region
$S_1 = 0 \Rightarrow P$ lies on the hyperbola



LINE AND A HYPERBOLA

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $c^2 >, =, < a^2m^2 - b^2$.

TANGENT AND NORMAL

Tangent :

Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(x_1\, y_1)$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y-y_1 = m_1(x-x_1)$ and $y-y_1=m_2(x-x_2)$, where m_1 and m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from (x_1, y_1) to the hyperbola.

Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Normal :

The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point $P(x_1,y_1)$ on the curve is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2=a^2e^2$.

The equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

In general, four normals can be drawn to a hyperbola from any point and if $\alpha, \beta, \gamma, \delta$ be the concentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ is an odd multiple of π.

Chord of Contact of Tangents Drawn from a Point Outside the Hyperbola

Chord of contact of tangents drawn from a point outside the hyperbola is $T = 0$ *i.e.*, $(xx_1/a^2) - (yy_1/b^2) = 1$.

CHORD OF HYPERBOLA WITH SPECIFIED MIDPOINT

Chord of hyperbola with specified midpoint (x_1, y_1) is $T = S_1$, where S_1 and T have usual meanings.

PAIR OF TANGENTS

Equation of pair of tangents from point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1 = T^2$ *i.e.*,

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents to a hyperbola is called its director circle. Its equation is $x^2 + y^2 = a^2 - b^2$.

Equation of any tangent to $x^2/a^2 - y^2/b^2 = 1$ is

$$y = mx \pm\sqrt{(a^2m^2-b^2)} \quad ....(i)$$

Tangent perpendicular to (i) is

$$y=-\frac{1}{m}x\pm\sqrt{[(a^2/m^2)-b^2]} \quad ....(ii)$$

Locus of point of intersection of these perpendicular tangents *i.e.*, equation of the director circle can be obtained by eliminating m between (i) and (ii).

$\therefore \quad (y - mx)^2 + (my + x)^2 = a^2m^2 - b^2 + a^2 - b^2m^2$

or $\quad (m^2 + 1)x^2 + (m^2 + 1)y^2 = (a^2 - b^2)(m^2 + 1)$

Cancelling $(m^2 + 1)$, we get the equation of director circle as $x^2 + y^2 = a^2 - b^2$.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. A straight line through $A(6, 8)$ meets the curve $2x^2 + y^2 = 2$ at B and C. P is a point on BC such that AB, AP, AC are in $H.P.$, then the minimum distance of the origin from the locus of 'P' is

(a) $\frac{1}{\sqrt{52}}$ (b) $\frac{5}{\sqrt{52}}$ (c) $\frac{10}{\sqrt{52}}$ (d) $\frac{15}{\sqrt{52}}$

2. Let $A(0, 2)$, B and C are points on parabola $y^2 = x + 4$ such that $\angle CBA = \frac{\pi}{2}$, then the range of ordinate of C is

(a) $(-\infty, 0) \cup (4, \infty)$ (b) $(-\infty, 0] \cup [4, \infty)$
(c) $[0, 4]$ (d) $(-\infty, 0) \cup [4, \infty)$

3. If $2p^2 - 3q^2 + 4pq - p = 0$ and a variable line $px + qy = 1$ always touches a parabola whose axis is parallel to x-axis, then equation of the parabola is

(a) $(y-4)^2 = 24(x-2)$
(b) $(y-3)^2 = 12(x-1)$
(c) $(y-4)^2 = 12(x-2)$
(d) $(y-2)^2 = 24(x-4)$

4. The locus of point of intersection of tangents to the parabola $y^2 = 4ax$, the angle between them being always 45° is

(a) $x^2 - y^2 + 6ax - a^2 = 0$
(b) $x^2 - y^2 - 6ax + a^2 = 0$
(c) $x^2 - y^2 + 6ax + a^2 = 0$
(d) $x^2 - y^2 - 6ax - a^2 = 0$

5. The locus of the vertex of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$, ($a$ is parameter) is

(a) $xy = \frac{105}{64}$ (b) $xy = \frac{3}{4}$
(c) $xy = \frac{35}{16}$ (d) $xy = \frac{64}{105}$

6. Tangents are drawn from the point $(-1, 2)$ to the parabola $y^2 = 4x$. The length of the intercept made by the line $x = 2$ on these tangents is

(a) 6 (b) $6\sqrt{2}$
(c) $2\sqrt{6}$ (d) None of these

7. The triangle PQR of area A is inscribed in the parabola $y^2 = 4ax$ such that P lies at the vertex of the parabola and base QR is a focal chord. The numerical difference of the ordinates of the points Q and R is

(a) $\frac{A}{2a}$ (b) $\frac{A}{a}$ (c) $\frac{2A}{a}$ (d) $\frac{4A}{a}$

8. If a variable tangent of the circle $x^2 + y^2 = 1$ intersects the ellipse $x^2 + 2y^2 = 4$ at P and Q, then the locus of the points of intersection of the tangents at P and Q is

(a) a circle of radius 2 units
(b) a parabola with focus as (2, 3)
(c) an ellipse with eccentricity $\frac{\sqrt{3}}{4}$
(d) an ellipse with length of latus rectum as 2 units

9. A circle $S = 0$ touches a circle $x^2 + y^2 - 4x + 6y - 23 = 0$ internally and the circle $x^2 + y^2 - 4x + 8y + 19 = 0$ externally. The locus of centre of the circle $S = 0$ is conic whose eccentricity is k then $\left[\frac{1}{k}\right]$ is, where $[\cdot]$ denotes G.I.F.

(a) 7 (b) 2 (c) 0 (d) 3

10. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with vertices having eccentric angles α, β, γ respectively is (x_1, y_1), then

$\Sigma\cos\alpha\cos\beta + \Sigma\sin\alpha\sin\beta =$

(a) $\frac{9x_1^2}{a^2} + \frac{9y_1^2}{b^2} + \frac{3}{2}$ (b) $9x_1^2 - 9y_1^2 + a^2b^2$
(c) $\frac{9x_1^2}{2a^2} + \frac{9y_1^2}{2b^2} - \frac{3}{2}$ (d) $\frac{9x_1^2}{a^2} + \frac{9y_1^2}{b^2} + 3$

11. The ratio of the area enclosed by the locus of mid-point of PS and area of the ellipse where P is any point on the ellipse and S is the focus of the ellipse, is

(a) 1 : 2 (b) 1 : 3 (c) 1 : 5 (d) 1 : 4

12. How many tangents to the circle $x^2 + y^2 = 3$ are there which are normal to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(a) 3 (b) 2 (c) 1 (d) 0

13. If the ellipse $\frac{x^2}{a^2 - 3} + \frac{y^2}{a + 4} = 1$ is inscribed in a square of side length $a\sqrt{2}$, then a is

(a) 4 (b) 2
(c) 1 (d) None of these

14. Let 'O' be the centre of ellipse for which A, B are end points of major axis and C, D are end points of minor axis, F is focus of the ellipse. If inradius of ΔOCF is 1, then $|AB| \times |CD| =$

(a) 65 (b) 52 (c) 78 (d) 47

15. The points of intersection of the two ellipse $x^2 + 2y^2 - 6x - 12y + 23 = 0$, $4x^2 + 2y^2 - 20x - 12y + 35 = 0$

(a) lie on a circle centred at $\left(\frac{8}{3}, 3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(b) lie on a circle centred at $\left(\frac{8}{3}, -3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{3}}$

(c) lie on a circle centred at (8,9) and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(d) are not concyclic.

16. In a model, it is shown that an arc of a bridge is semi elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal, then the height of the arch, 2 m from the centre of the base is (in meters)

(a) $\frac{8}{3}$ (b) $\frac{\sqrt{65}}{3}$ (c) $\frac{\sqrt{56}}{3}$ (d) $\frac{9}{3}$

17. If a chord joining $P(a \sec\theta, a \tan\theta)$, $Q(a \sec\alpha, a \tan\alpha)$ on the hyperbola $x^2 - y^2 = a^2$ is normal at P, then $\tan\alpha =$

(a) $\tan\theta\,(4\sec^2\theta + 1)$ (b) $\tan\theta\,(4\sec^2\theta - 1)$
(c) $\tan\theta\,(2\sec^2\theta - 1)$ (d) $\tan\theta\,(1 - 2\sec^2\theta)$

18. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2 is

(a) $16x^2 + 10xy + y^2 = 2$
(b) $16x^2 - 10xy + y^2 = 2$
(c) $16x^2 + 10xy + y^2 = 4$
(d) $16x^2 - 10xy + y^2 = 4$

19. Which of the following is incorrect for the hyperbola $x^2 - 2y^2 - 2x + 8y - 1 = 0$

(a) Its eccentricity is $\sqrt{2}$
(b) Length of the transverse axis is $2\sqrt{3}$
(c) Length of the conjugate axis is $2\sqrt{6}$
(d) Latus rectum is $4\sqrt{3}$

20. The area of the triangle formed by the asymptotes and any tangent to the hyperbola $x^2 - y^2 = a^2$ is

(a) $4a^2$ (b) $3a^2$ (c) $2a^2$ (d) a^2

21. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is

(a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$

22. Equation of a common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = x + 8$ (d) $y = x + 2$

23. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then the eccentricity e of the hyperbola, satisfies

(a) $1 < e < 2/\sqrt{3}$ (b) $e = 2/\sqrt{3}$
(c) $e = \sqrt{3}/2$ (d) $e > 2/\sqrt{3}$

24. The locus of a point, from where tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ makes an angle of 45°, is

(a) $(x^2 + y^2) + a^2(x^2 - y^2) = 4a^2$
(b) $2(x^2 + y^2) + 4a^2(x^2 - y^2) = 4a^2$
(c) $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^4$
(d) $(x^2 + y^2)^2 + a^2(x^2 - y^2) = a^4$

25. If a circle cuts the rectangular hyperbola $xy = 1$ in 4 points (x_r, y_r) where $r = 1, 2, 3, 4$. Then orthocentre of triangle with vertices at $(x_r, y_r)$where $r = 1, 2, 3$ is

(a) (x_4, y_4) (b) $(-x_4, -y_4)$
(c) $(-x_4, y_4)$ (d) $(x_4, -y_4)$

SECTION-II

Multiple Correct Answer Type

26. If PQ and RS are normal chords of the parabola $y^2 = 8x$ and the points P, Q, R, S are concyclic then

(a) Tangents at P and R meet on x-axis
(b) Tangents at P and R meet on y-axis
(c) PR is parallel to y-axis
(d) PR is parallel to x-axis

27. $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, $C(at_3^2, 2at_3)$ be 3 points on the parabola $y^2 = 4ax$. If the orthocentre of ΔABC is focus S of the parabola, then

(a) $t_1t_2 + t_3t_2 + t_3t_1 = -5$
(b) $\frac{1}{t_1t_2} + \frac{1}{t_2t_3} + \frac{1}{t_3t_1} = -1$
(c) If $t_1 = 0$, then $t_2 + t_3 = 0$
(d) $(1 + t_1)(1 + t_2)(1 + t_3) = -4$

28. Consider the parabola represented by the parametric equations $x = t^2 - 2t + 2$, $y = t^2 + 2t + 2$. Then which of the following is/are true?
(a) Auxiliary circle of the parabola is $x^2 + y^2 = 4$
(b) Vertex of the parabola is (2, 2)
(c) Director circle of the parabola is $x^2 + y^2 = 6$
(d) Focus of the parabola is (3, 3)

29. The equations of the common tangents of the curves $x^2 + 4y^2 = 8$ and $y^2 = 4x$ are
(a) $x + 2y + 4 = 0$ (b) $x - 2y + 4 = 0$
(c) $2x + y = 4$ (d) $2x - y + 4 = 0$

30. Let PQ be a chord of the parabola $y^2 = 4x$. A circle is drawn with PQ as diameter passes through the vertex 'V' of the parabola. If area of triangle PVQ is 20 sq. units, then the coordinates of P are
(a) (16, 8) (b) (16, –8)
(c) (–16, 8) (d) (–16, –8)

31. If $Ay^2 + By + Cx + D = 0$, $(ABC \neq 0)$ be the equation of parabola, then
(a) the length of latusrectum is $\left|\frac{C}{A}\right|$
(b) the axis of the parabola is a vertical line
(c) y-coordinate of the vertex is $-\frac{B}{2A}$
(d) x-coordinate of the vertex is $\left(\frac{B^2 - 4AD}{4AC}\right)$

32. An ellipse whose major axis is parallel to x-axis is such that the segments of a focal chord are 1 and 3 units. The lines $ax + by + c = 0$ are the chords of the ellipse such that a, b, c are in A.P. and bisected by the point at which they are concurrent. The equation of auxiliary circle is $x^2 + y^2 + 2\alpha x + 2\beta y - 2\alpha - 1 = 0$. Then
(a) The locus of perpendicular tangents to the ellipse is $x^2 + y^2 = 7$
(b) Length of the double ordinate which is conjugate to directrix is 3
(c) Area of the auxiliary circle is 2π
(d) Eccentricity of the ellipse is $\frac{1}{2}$

33. If foci of an ellipse be (–1, 2) and (–2, 3) and its tangent at a point A is $2x + 3y + 9 = 0$, then
(a) Length of the minor axis of the ellipse will be $2\sqrt{14}$
(b) Coordinates of the point 'A' will be $\left(\frac{-32}{9}, \frac{-17}{27}\right)$
(c) Distance between the foci is $2\sqrt{2}$
(d) Product of the perpendiculars from foci to any tangent is 56

34. Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $y_1 > 0$, $y_2 > 0$ be the end points of the latus rectum of the ellipse $3x^2 + 4y^2 = 12$. The equations of the parabolas with latus rectum PQ are
(a) $x^2 - 2y - 2 = 0$ (b) $x^2 - 2y + 2 = 0$
(c) $x^2 + 2y - 4 = 0$ (d) $x^2 - 2y + 4 = 0$

35. If the chord through the points whose eccentric angles are θ and ϕ on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ passes through a focus, then $\tan\frac{\theta}{2} \cdot \tan\frac{\phi}{2}$ is
(a) $\frac{1}{9}$ (b) –9 (c) $-\frac{1}{9}$ (d) 9

36. If a tangent of slope $\frac{1}{3}$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, (a > b)$ is normal to the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, then
(a) maximum value of ab is $\frac{2}{3}$
(b) $a \in \left(\sqrt{\frac{2}{5}}, 2\right)$
(c) $a \in \left(\frac{2}{3}, 2\right)$
(d) maximum value of ab is 1

37. A point on the ellipse $x^2 + 3y^2 = 37$ where the normal is parallel to the line $6x - 5y = 2$ is
(a) (5, –2) (b) (5, 2)
(c) (–5, 2) (d) (–5, –2)

38. If the normal at P to the rectangular hyperbola meets the axes in G and g, and C is centre of the hyperbola, then
(a) $PG = PC$ (b) $Pg = PC$
(c) $PG = Pg$ (d) $Gg = 2PC$

39. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn

(a) If $n = 1$, then $e = \sqrt{2}$

(b) If $n = 0$, then $e < \sqrt{2}$

(c) If $n > 1$, then $0 < e < \sqrt{2}$

(d) None of these

40. A rectangular hyperbola of latus rectum 4 units passes through (0, 0) and has (2, 0) as its one focus. The equation of locus of the other focus is

(a) $x^2 + y^2 = 36$ (b) $x^2 + y^2 = 4$

(c) $x^2 - y^2 = 4$ (d) $x^2 + y^2 = 9$

41. The equation of tangent to the hyperbola $5x^2 - y^2 = 5$ passing through the point (2, 8) is/are

(a) $3x - y + 2 = 0$ (b) $3x + y - 14 = 0$

(c) $23x - 3y - 22 = 0$ (d) $3x - 23y + 178 = 0$

SECTION-III

Comprehension Type

Paragraph for Question No. 42 to 43

Let $R(h, k)$ be the middle point of the chord PQ of the parabola $y^2 = 4ax$, then its equation will be $ky - 2ax + 2ah - k^2 = 0$. The locus of the mid-point of chords of the parabola which

42. Subtend a constant angle α at the vertex is $(y^2 - 2ax + 8a^2)^2 \tan^2\alpha = \lambda a^2(4ax - y^2)$, where $\lambda =$

(a) 4 (b) 8 (c) 16 (d) 32

43. Are such that the focal distances of their extremities are in the ratio 2: 1 is $9(y^2 - 2ax)^2 = \lambda a^2(2x - a)(4x + a)$, where $\lambda =$

(a) 4 (b) 8 (c) 16 (d) 12

Paragraph for Question No. 44 to 46

The normal at any point (x_1, y_1) of curve is a line perpendicular to tangent at the point (x_1, y_1). In case of parabola $y^2 = 4ax$, the equation of normal is $y = mx - 2am - am^3$ (m is slope of normal). In case of rectangular hyperbola $xy = c^2$, the equation of normal at $(ct, c/t)$ is $xt^3 - yt - ct^4 + c = 0$. The shortest distance between any two curves always exist along the common normal.

44. If normal at (5, 3) of rectangular hyperbola $xy - y - 2x - 2 = 0$ intersect it again at a point

(a) (–1, 0) (b) (–1, 1)

(c) (0, –2) (d) (3/4, –14)

45. The shortest distance between the parabolas $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ is

(a) $2\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) 4 (d) $\sqrt{\frac{36}{5}}$

46. Number of normals drawn from $\left(\frac{7}{6}, 4\right)$ to parabola $y^2 = 2x - 1$ is

(a) 1 (b) 2 (c) 3 (d) 4

Paragraph for Question No. 47 to 49

Conic possesses enormous properties which can be proved by taking their standard forms. Unlike circle these properties rarely follow geometrical considerations. Most of the properties of conic are proved analytically. For example, the properties of a parabola can be proved by taking its standard equation $y^2 = 4ax$ and a point $(at^2, 2at)$ on it

47. If the tangent and normal at any point 'P' on the parabola whose focus is S, meets its axis in T and G respectively, then

(a) $PG = GT$ (b) S is mid-point of T and G

(c) $ST = 2SG$ (d) None of these

48. The angle between the tangents drawn at the extremities of a focal chord must be

(a) 30° (b) 60° (c) 90° (d) 120°

49. If the tangent at any point 'P' meets the directrix at K, then $\angle KSP$ must be

(a) 30° (b) 60°

(c) 90° (d) None of these

Paragraph for Question No. 50 to 51

A sequence of ellipse $E_1, E_2 ... E_n$ are constructed as follows:

Ellipse E_n is drawn so as to touch ellipse E_{n-1} at the extremities of the major axis of E_{n-1} and have its foci at the extremities of the minor axis of E_{n-1}.

50. If E_n is independent of n, then eccentricity of the ellipse E_{n-2} is

(a) $\frac{3-\sqrt{5}}{2}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\frac{2-\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}-1}{2}$

51. If eccentricity of ellipse E_n is independent of n, then the locus of the mid-point of chords of slope –1 of E_n (If the axis of E_n is along y-axis)

(a) $(\sqrt{5}-1)x = 2y$ (b) $(\sqrt{5}+1)x = 2y$

(c) $(3-\sqrt{5})x = 2y$ (d) $(3+\sqrt{5})x = 2y$

Paragraph for Question No. 52 to 54

$C_1 : x^2 + y^2 = r_2$ and $C_2 : \frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four

distinct points A, B, C and D. Their common tangents form a parallelogram $A'B'C'D'$.

52. If $ABCD$ is a square, then r is equal to

(a) $\frac{12}{5}\sqrt{2}$ (b) $\frac{12}{5}$

(c) $\frac{12}{5\sqrt{5}}$ (d) None of these

53. If $A'B'C'D'$ is a square, then r is equal to

(a) $\sqrt{20}$ (b) $\sqrt{12}$

(c) $\sqrt{15}$ (d) None of these

54. If $A'B'C'D'$ is a square, then the ratio of area of the circle C_1 to the area of the circumcircle of $\Delta A'B'C'$ is

(a) $\frac{9}{16}$ (b) $\frac{3}{4}$

(c) $\frac{1}{2}$ (d) None of these

Paragraph for Question No. 55 to 56

If $H : x^2 - y^2 = 9$; $P : y^2 = 4(x - 5)$, $L : x = 9$

55. If L is the chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 + 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x + 9 = 0$
(d) $9x^2 - 8y^2 - 18x - 9 = 0$

56. If R is the point of intersection of the tangents to H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$ (c) $y = 7$ (d) $y = 9$

Paragraph for Question No. 57 to 58

In hyperbola portion of tangent intercept between asymptotes is bisected at the point of contact. Consider a hyperbola whose centre is at origin. A line $x + y = 2$ touches this hyperbola at $P(1,1)$ and intersects the asymptotes at A and B such that $AB = 6\sqrt{2}$ units.

57. Equation of asymptotes are

(a) $5xy + 2x^2 + 2y^2 = 0$ (b) $3x^2 + 2y^2 + 6xy = 0$
(c) $2x^2 + 2y^2 - 5xy = 0$ (d) $2x^2 + 3x^2 + 5x = 0$

58. Equation of the tangent to the hyperbola at $\left(-1, \frac{7}{2}\right)$ is

(a) $5x + 2y = 2$ (b) $3x + 2y = 4$
(c) $3x + 4y = 11$ (d) $3x + 2y = 6$

SECTION-IV

Matrix-Match Type

59. Consider the parabola

$$(x-1)^2 + (y-2)^2 = \frac{(12x-5y+3)^2}{169}$$

	Column - I		Column - II
(A)	Locus of point of intersection of perpendicular tangent	(p)	$12x - 5y - 2 = 0$
(B)	Locus of foot of perpendicular from focus upon any tangent	(q)	$5x + 12y - 29 = 0$
(C)	Line along which minimum length of focal chord occurs	(r)	$12x - 5y + 3 = 0$
(D)	Line about which parabola is symmetrical	(s)	$24x - 10y + 1 = 0$

60. Match the following

	Column - I		Column - II
(A)	If the distances of two points P and Q lie on the parabola $y^2 = 4ax$ from the focus S of the same parabola are 4 and 9 respectively, then the distance of the point of intersection R of tangents at P and Q from the focus is equal to	(p)	8
(B)	The normal chord of a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus, is cosec^{-1} (?)	(q)	4
(C)	The distance between a tangent to the parabola $y^2 = 4ax$ $(a > 0)$ and parallel normal with gradient 1, is $\sqrt{pa}$,then $p =$	(r)	6
(D)	If the normal to a parabola $y^2 = 4ax$ at P meets the curve again at Q and if PQ and the normal at Q makes angles α and β respectively, then $\lvert 2\tan\alpha(\tan\alpha + \tan\beta)\rvert$ equals to	(s)	1

61. Match the following

	Column - I		Column - II
(A)	The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is	(p)	7
(B)	The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then b^2 is	(q)	0
(C)	The product of lengths of perpendiculars from any point t of the hyperbola $x^2 - y^2 = 8$ to its asymptotes is	(r)	$\sqrt{2}$
(D)	The number of points outside the hyperbola $\frac{x^2}{25}-\frac{y^2}{36}=1$ from where two perpendicular tangents can be drawn to the hyperbola is/are	(s)	1

62. Match the following

	Column - I		Column - II
(A)	The coordinates of the point on the parabola $y = x^2 + 7x + 2$, which is nearest to the straight line $y = 3x - 3$ are	(p)	(2,1)
(B)	$y = x + 2$ is a tangent to the parabola $y^2 = 8x$. The point on this line, the other tangent from which is perpendicular to this tangent is	(q)	$(-2, 0)$
(C)	The point on the ellipse $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is least is	(r)	$\left(2, \frac{2}{\sqrt{3}}\right)$
(D)	The foci of the ellipse $\frac{x^2}{25}+\frac{h^2}{9}=1$ are S and S'. P is a point on the ellipse whose eccentric angle is $\pi/3$. The incentre of the triangle SPS' is	(s)	$(-2, -8)$
		(t)	(2, 2)

63. Match the following

	Column - I		Column - II
(A)	The length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is	(p)	$\frac{2}{3}$
(B)	The product of the perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 = 2$ to its asymptotes is	(q)	3
(C)	The length of the transverse axis of the hyperbola $xy = 18$ is	(r)	$\frac{32}{3}$
(D)	The product of the lengths of the perpendiculars drawn from the foci of $3x^2 - 4y^2 = 12$ on any of its tangents is	(s)	12

SECTION-V

Integer Answer Type

64. Consider the parabola $y^2 = 4x$. Let P and Q be two points $(4, -4)$ and $(9, 6)$ respectively on the parabola. Let R be a moving point on the arc of the parabola between P and Q. If the maximum area of ΔRPQ is 'S', then $(4S)^{\frac{1}{3}}$ equals

65. Two tangents are drawn from point (1, 4) to the parabola $y^2 = 4x$. Angle between these tangents is $\frac{\pi}{K}$, then $K =$

66. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then $k(> 0)$ is

67. If $x + y = k$ is normal to $y^2 = 12x$, then k is

68. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{\pi}{3}$. Then the eccentricity of conjugate hyperbola is

69. The distance between the directrices of the ellipse $(4x-8)^2+16y^2=(x+\sqrt{3}y+10)^2$ is K, then $\frac{K}{2}$ is

70. Number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

71. If L be the length of common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ is intercepted by the coordinate axis, then $\frac{\sqrt{3}L}{2}$ is

72. If P and Q are the ends of a pair of conjugate diameters and C is the centre of the ellipse $4x^2 + 9y^2 = 36$, then the area of ΔCPQ in square units is

73. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

74. Acute angle between the asymptotes of the hyperbola $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$ is θ. Then $\tan\theta =$

SOLUTIONS

1. **(a):** $(6 + r\cos\theta, 8 + r\sin\theta)$ lies on $2x^2 + y^2 = 2$

$\Rightarrow$ $(2\cos^2\theta + \sin^2\theta)r^2 + 2(12\cos\theta + 8\sin\theta)r + 134 = 0$

AB, AP, AC are in H.P. $\Rightarrow \frac{2}{r}=\frac{AB+AC}{AB\cdot AC}$

$\Rightarrow \frac{1}{r}=-\frac{(6\cos\theta+4\sin\theta)}{67}\Rightarrow 6x+4y-1=0$

Minimum distance from 'O' $=\frac{1}{\sqrt{52}}$

2. **(b):** $A(0, 2), B(t_1^2 - 4, t_1), C(t^2 - 4, t)$

$\Rightarrow \frac{2-t_1}{4-t_1^2}\cdot\frac{t_1-t}{t_1^2-t^2}=-1\Rightarrow\frac{1}{2+t_1}\cdot\frac{1}{t+t_1}=-1$

$\Rightarrow t_1^2 + (2 + t)t_1 + (2t + 1) = 0$

For real t_1, $(2 + t)^2 - 4(2t + 1) \ge 0 \Rightarrow t^2 - 4t \ge 0$

$\Rightarrow t \in (-\infty, 0]) \cup [4, \infty)$

3. **(c):** The parabola be $(y - a)^2 = 4b(x - c)$

Equation of tangent is $(y-a)=-\frac{p}{q}(x-c)-\frac{bq}{p}$

Comparing with $px + qy = 1$, we get

$cp^2 - bq^2 + apq - p = 0$

$\therefore \frac{c}{2}=\frac{b}{3}=\frac{a}{4}=1$

$\Rightarrow$ The equation is $(y - 4)^2 = 12(x - 2)$

4. **(c):** Equation of tangent is $y=mx+\frac{a}{m}$

$\Rightarrow m^2x - my + a = 0 \Rightarrow m_1+m_2=\frac{y}{x}, m_1m_2=\frac{a}{x}$

$\tan 45^\circ=\left|\frac{m_1-m_2}{1+m_1m_2}\right|\Rightarrow\left(\frac{y}{x}\right)^2-4\left(\frac{a}{x}\right)=\left(1+\frac{a}{x}\right)^2$

$\Rightarrow x^2 - y^2 + 6ax + a^2 = 0$

5. **(a):** $y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$

$y=\frac{2a^3}{6}\left(x^2+\frac{3}{2a}x-\frac{12a}{2a^3}\right)$

$y=\frac{2a^3}{6}\left(x^2+2.\frac{3}{4a}x+\frac{9}{16a^2}-\frac{9}{16a^2}-\frac{12a}{2a^3}\right)$

$y=\frac{2a^3}{6}\left(\left(x+\frac{3}{4a}\right)^2-\frac{105a}{16a^3}\right)$

$\left(y+\frac{105a}{48}\right)=\frac{2a^3}{6}\left(x+\frac{3}{4a}\right)^2$

$y=\frac{-105a}{48}$, $x=\frac{-3}{4a}$

$yx=\frac{105a}{48}\times\frac{3}{4a}=\frac{105}{64}$

6. **(b):** Equation of pair of tangent is

$SS_1 = T^2$

$\Rightarrow (y^2 - 4x)(8) = 4(y - x + 1)^2$

$\Rightarrow y^2 - 2y(1 - x) - (x^2 + 6x + 1) = 0$

Put $x = 2$

$\Rightarrow y^2 + 2y - 17 = 0$

$\Rightarrow |y_1-y_2|=6\sqrt{2}$

7. **(c):** QR is a focal chord

$\Rightarrow R(at^2, 2at)$ and $Q\left(\frac{a}{t^2},-\frac{2a}{t}\right)$

$\Rightarrow d=\left|2at+\frac{2a}{t}\right|=2a\left|t+\frac{1}{t}\right|$

Now $A=\frac{1}{2}\begin{vmatrix}at^2 & 2at & 1\\ \frac{a}{t^2} & -\frac{2a}{t} & 1\\ 0 & 0 & 1\end{vmatrix}=a^2\left|t+\frac{1}{t}\right|$

$\Rightarrow 2a\left|t+\frac{1}{t}\right|=\frac{2A}{a}$

8. **(d):** $x^2 + y^2 = 1$, $x^2 + 2y^2 = 4$

Let $R(x_1, y_1)$ be point of intersection of tangents drawn at P, Q to ellipse

$\Rightarrow$ PQ is chord of contact of $R(x_1, y_1)$

$\Rightarrow xx_1 + 2yy_1 - 4 = 0$

This touches circle $\Rightarrow r^2(l^2+m^2)=n^2$

$\Rightarrow \quad 1(x_1^2+4y_1^2)=16$

$\Rightarrow \quad x^2+4y^2=16$ is ellipse with $e=\dfrac{\sqrt{3}}{2}$;

Length of latus rectum = 2 units

9. (a): $c_1(2,-3)r_1=6$

$c_2(2,-4)r_2=1$

Let c is the centre and r be the radius of $S=0$

$$\left.\begin{aligned} cc_1 &= r_1-r \\ cc_2 &= r_2+r \end{aligned}\right\} \Rightarrow cc_1+cc_2=r_1+r_2$$

$\therefore$

$\therefore$ Locus is an ellipse whose foci are $(2,-3)$ and $(2,-4)$

$$e=\frac{2ae}{2a}=\frac{c_1c_2}{r_1+r_2}=\frac{1}{7}\Rightarrow k=\frac{1}{7}$$

10. (c): $(x_1,y_1)=\left(\dfrac{a\sum\cos\alpha}{3},\dfrac{b\sum\sin\alpha}{3}\right)$

$$\sum\cos\alpha=\frac{3x_1}{a} \qquad \text{....(1)}$$

$$\sum\sin\alpha=\frac{3y_1}{b} \qquad \text{....(2)}$$

Squaring and adding, we get the answer.

11. (d): Equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, Area = πab

Let $P=(a\cos\theta, b\sin\theta)$

$S=(ae,0)$

$M(h,k)$ be the mid-point of PS

$$\Rightarrow \quad h=\frac{ae+a\cos\theta}{2};k=\frac{b\sin\theta}{2}\Rightarrow\frac{\left(h-\frac{ae}{2}\right)^2}{(a^2/4)}+\frac{k^2}{(b^2/4)}=1$$

Locus of (h,k) is ellipse

$$\text{Area} = \pi\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)=\frac{1}{4}\pi ab$$

12. (d): Equation of normal at $P(3\cos\theta, 2\sin\theta)$ is $3x\sec\theta-2y\,\mathrm{cosec}\,\theta=5$

$$\frac{5}{\sqrt{9\sec^2\theta+4\,\mathrm{cosec}^2\theta}}=\sqrt{3}$$

But min. of $9\sec^2\theta+4\,\mathrm{cosec}^2\theta=25$

$\therefore$ No such θ exists.

13. (d): Sides of the square will be perpendicular tangents to the ellipse. So, vertices of the square will lie on director circle. So diameter of director circle is

$$2\sqrt{(a^2-3)+(a+4)}=\sqrt{2a^2+2a^2}$$

$$2\sqrt{a^2+a+1}=2a\Rightarrow a=-1$$

But for ellipse $a^2>3$ and $a>-4$

So a cannot take the value '–1'.

14. (a)

15. (a): If $S_1=0$ and $S_2=0$ are the equations, then $\lambda S_1+S_2=0$ is a second degree curve passing through the points of intersection of $S_1=0$ and $S_2=0$

$$\Rightarrow \quad (\lambda+4)x^2+2(\lambda+1)y^2-2(3\lambda+10)x$$
$$-12(\lambda+1)y+(23\lambda+35)=0$$

For it to be a circle, choose λ such that the coefficients of x^2 and y^2 are equal.

$\therefore \quad \lambda=2$

This gives the equation of the circle as

$$6(x^2+y^2)-32x-36y+81=0$$

$$\Rightarrow \quad x^2+y^2-\frac{16}{3}x-6y+\frac{27}{2}=0$$

Its centre is $C\left(\dfrac{8}{3},3\right)$ and radius is

$$r=\sqrt{\frac{64}{9}+9-\frac{27}{2}}=\frac{1}{3}\sqrt{\frac{47}{2}}$$

16. (b): Let the equation of the semi elliptical arc be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(y>0)$$

Length of the major axis $=2a=9\Rightarrow a=9/2$

So the equation of the arc becomes $\dfrac{4x^2}{81}+\dfrac{y^2}{9}=1$

If $x=2$, then $y^2=\dfrac{65}{9}\Rightarrow y=\dfrac{1}{3}\sqrt{65}$

17. (b): Slope of chord joining P and Q = slope of normal at P

$$\frac{\tan\alpha-\tan\theta}{\sec\alpha-\sec\theta}=-\frac{\tan\theta}{\sec\theta}\Rightarrow\tan\alpha-\tan\theta=-k\tan\theta$$

and $\sec\alpha-\sec\theta=k\sec\theta$

$\therefore (1-k)\tan\theta=\tan\alpha$ and $(1+k)\sec\theta=\sec\alpha$

$[(1+k)\sec\theta]^2-[(1-k)\tan\theta]^2=\sec^2\alpha-\tan^2\alpha$

$\Rightarrow \quad k=-2(\sec^2\theta+\tan^2\theta)=-4\sec^2\theta+2$

$\tan\alpha=\tan\theta(1+4\sec^2\theta-2)=\tan\theta(4\sec^2\theta-1)$

18. (a): Let $P(h,k)$

$y-k=4(x-h)$

Let it meets $xy=1$ at $A(x_1,y_1)$ and $B(x_2,y_2)$

$\therefore \quad x_1 + x_2 = \frac{4h-k}{4}, x_1x_2 = -\frac{1}{4}$

Also $\frac{2x_1 + x_2}{3} = h \Rightarrow x_1 = \frac{8h+k}{4}, x_2 = -\frac{(2h+k)}{2}$

$\Rightarrow \quad 16x^2 + 10xy + y^2 = 2$

19. (a): The equation of the hyperbola is

$x^2 - 2y^2 - 2x + 8y - 1 = 0$

or $\quad (x-1)^2 - 2(y-2)^2 + 6 = 0$

or $\quad \frac{(x-1)^2}{-6} + \frac{(y-2)^2}{3} = 1$

or $\quad \frac{(y-2)^2}{3} - \frac{(x-1)^2}{6} = 1 \quad$ ...(1)

or $\quad \frac{Y^2}{3} - \frac{X^2}{6} = 1, \quad$ ...(2)

[where $X = x - 1$ and $Y = y - 2$]

$\therefore$ The centre = (0, 0) in the X-Y coordinates

$\therefore$ The centre = (1, 2) in the x-y coordinates using (2)

If the transverse axis be of length $2a$, then $a = \sqrt{3}$, since in the equation (1) the transverse axis is parallel to the y-axis. If the conjugate axis is of length $2b$, then $b = \sqrt{6}$.

But $b^2 = a^2 (e^2 - 1)$

$\therefore \quad 6 = 3(e^2 - 1) \Rightarrow e^2 = 3$ or $e = \sqrt{3}$

The length of the transverse axis = $2\sqrt{3}$

The length of the conjugate axis = $2\sqrt{6}$

Length of latus rectum = $4\sqrt{3}$

20. (d): Equation of any tangent to $x^2 - y^2 = a^2$

i.e. $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ is $\frac{x}{a}\sec\theta - \frac{y}{a}\tan\theta = 1 \quad$ ....(1)

or $\quad x \sec\theta - y \tan\theta = a$

The two asymptotes of the hyperbola $x^2 - y^2 = a^2$ are $x - y = 0$ and $x + y = 0$

$\therefore$ Equation of other two sides of the triangle are

$x - y = 0 \quad$ ....(2)

$x + y = 0 \quad$ ....(3)

Solving (1), (2) and (3) in pairs, the coordinates of the vertices of the triangle are (0, 0)

$\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right)$

and $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right)$

Area of triangle $= \frac{1}{2}\left|\frac{-a^2}{\sec^2\theta - \tan^2\theta} - \frac{a^2}{\sec^2\theta - \tan^2\theta}\right|$

$= \frac{1}{2}\left(a^2 + a^2\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$= a^2$

21. (b): Normal at (6, 3) is

$\frac{a^2x}{6} + \frac{b^2y}{3} = a^2 + b^2$

$\because$ It passes through (9, 0) also

$\Rightarrow \quad \frac{9a^2}{6} = a^2 + b^2 \Rightarrow \frac{3}{2} = 1 + \frac{b^2}{a^2}$

$\therefore \quad \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow e^2 - 1 = \frac{1}{2} \Rightarrow e = \sqrt{\frac{3}{2}}$

22. (d): $y^2 = 8x$, $xy = -1$

Let $P\left(t, \frac{-1}{t}\right)$ be any point on $xy = -1$

Equation of the tangent to $xy = -1$ at $P\left(t, \frac{-1}{t}\right)$ is

$\frac{-x}{t} + yt = -2$

$y = \frac{x}{t^2} + \left(\frac{-2}{t}\right) \quad$ ....(1)

If (1) is tangent to the parabola $y^2 = 8x$, then

$\frac{-2}{t} = \frac{2}{1/t^2} \Rightarrow t^3 = -1$

$\Rightarrow \quad t = -1$

$\therefore \quad$ Common tangent is $y = x + 2$

23. (d): If OPQ is equilateral triangle, then OP makes 30° with x-axis.

$\left(\frac{\sqrt{3}r}{2}, \frac{r}{2}\right)$, lies on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \quad r^2 = \frac{16a^2b^2}{12b^2 - 4a^2} > 0$

$\Rightarrow \quad 12b^2 - 4a^2 > 0 \Rightarrow \frac{b^2}{a^2} > \frac{4}{12}$

$\Rightarrow \quad e^2 - 1 > \frac{1}{3}$

$\Rightarrow \quad e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

24. (c) : Equation of tangent to the hyperbola is

$$y = mx \pm \sqrt{m^2a^2 - a^2}$$

Let $P(x_1, y_1)$ be the point

$$y_1 - mx_1 = \pm\sqrt{m^2a^2 - a^2}$$

S.B.S.

$\Rightarrow\ m^2(x_1^2 - a^2) - 2\,y_1x_1m + y_1^2 + a^2 = 0$

$\therefore\ m_1 + m_2 = \dfrac{2x_1y_1}{x_1^2 - a^2},\ m_1m_2 = \dfrac{y_1^2 + a^2}{x_1^2 - a^2}$

$$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$\Rightarrow\ (1 + m_1m_2)^2 = (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4\,m_1m_2$

$\Rightarrow\ \left(1 + \dfrac{y_1^2 + a^2}{x_1^2 - a^2}\right)^2 = \left(\dfrac{2x_1y_1}{x_1^2 - a^2}\right)^2 - 4\left(\dfrac{y_1^2 + a^2}{x_1^2 - a^2}\right)$

25. (b): $xy = 1$ cuts the circle in 4 points, then $x_1x_2x_3x_4 = 1,\ y_1y_2y_3y_4 = 1$

Orthocentre of triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3)

i.e., $\left(\dfrac{-1}{x_1x_2x_3}, -(y_1y_2y_3)^{-1}\right)$ *i.e.* $(-x_4, -y_4)$

26. (a, c): Equation of normal chords at $P(2t_1^2, 4t_1)$ and $R(2t_2^2, 4t_2)$ are $y + t_1x - 4t_1 - 2t_1^3 = 0$ and $y + t_2x - 4t_2 - 2t_2^3 = 0$

Equation of curve through P, Q, R, S is

$(y + t_1x - 4t_1 - 2t_1^3)\,(y + t_2x - 4t_2 - 2t_2^3) + \lambda(y^2 - 8x) = 0$

P, Q, R, S are concyclic, $t_1 + t_2 = 0$ and $t_1\,t_2 = 1 + \lambda$

Points of intersection of tangents $(at_1t_2, a(t_1 + t_2))$ lies on x-axis, slope of $PR = \dfrac{z}{t_1 + t_2}$

$\therefore$ PR is parallel to y-axis.

27. (a, b, c): Slope of $AS = \dfrac{2at_1}{at_1^2 - a} = \dfrac{2t_1}{t_1^2 - 1}$

Slope of $BC = \dfrac{2a(t_3 - t_2)}{a(t_3^2 - t_2^2)} = \dfrac{2}{t_3 + t_2}$

$\therefore\ \dfrac{2t_1}{t_1^2 - 1} \times \dfrac{2}{t_3 + t_2} = -1$

$\Rightarrow\ -4t_1 = t_3t_1^2 + t_1^2t_2 - t_3 - t_2$

If $t_1 = 0 \Rightarrow t_3 + t_2 = 0$

So, option(c) is correct.

$t_1^2(t_2 + t_3) + 4t_1 = t_3 + t_2$ ...(1)

Similarly $t_2^2(t_1 + t_3) + 4t_2 = t_1 + t_3$ ...(2)

On subtracting (2) from (1), we get

$t_1^2t_2 + t_1^2t_3 - t_2^2t_1 - t_2^2t_3 + 4(t_1 - t_2) = t_2 - t_1$

$t_1t_2(t_1 - t_2) + t_3(t_1^2 - t_2^2) = 5(t_2 - t_1)$

$t_1t_2 + t_3\,(t_1 + t_2) = -5$

$\therefore\ \sum t_1t_2 = -5$

So, option (a) is true

Now, $t_1^2(t_2 + t_3) + 4t_1 = t_3 + t_2$

$t_1(t_1t_2 + t_3t_1) + 4t_1 = t_2 + t_3$

$t_1(-5 - t_2t_3) + 4t_1 = t_2 + t_3$

$-t_1t_2t_3 - t_1 = t_2 + t_3$

$-t_1t_2t_3 = t_1 + t_2 + t_3$

$\therefore\ \dfrac{1}{t_1t_2} + \dfrac{1}{t_2t_3} + \dfrac{1}{t_3t_1} = -1$

So, (b) is correct.

28. (a, b, d) **29.** (a, b) **30.** (a, b) **31.** (a, c, d)
32. (b, d) **33.** (a, b) **34.** (b, c) **35.** (b, c)
36. (a, b) **37.** (b, d) **38.** (a, b, c, d)
39. (a, c) **40.** (a) **41.** (a, c) **42.** (c)
43. (a) **44.** (d) **45.** (b) **46.** (a) **47.** (b)
48. (c) **49.** (c) **50.** (b) **51.** (b) **52.** (a)
53. (d) **54.** (c) **55.** (c) **56.** (b) **57.** (a)
58. (b) **59.** A → r; B → s; C → p; D → q
60. A → r; B → s; C → p; D → q
61. A → r; B → p; C → s; D → q
62. A → s; B → q; C → p; D → r
63. A → r; B → p; C → s; D → q
64. (5) **65.** (3) **66.** (4) **67.** (9) **68.** (2)
69. (8) **70.** (4) **71.** (7) **72.** (3) **73.** (4)
74. (2)

CONCEPT BOOSTERS

Class XII

VECTORS AND THREE DIMENSIONAL GEOMETRY

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

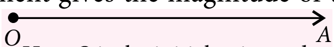
DEFINITION

A scalar is a quantity, which has only magnitude but does not have any direction. For example time, mass, temperature, specific gravity etc. are scalars.

A vector is a quantity which has magnitude as well as direction. For example displacement, force, acceleration are vectors.

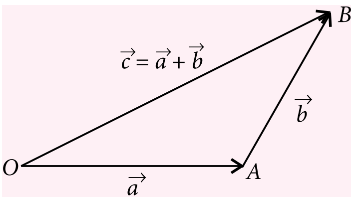
(a) There are different ways of denoting a vector : $\vec{a}$ or $\bar{a}$ or ***a***. We use for our convenience $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. to denote vectors and a, b, c to denote their magnitudes. Magnitude of a vector $\vec{a}$ is also written as $|\vec{a}|$.

(b) A vector $\vec{a}$ may be represented by a line segment OA and arrow gives direction of this vector. Length of the line segment gives the magnitude of the vector.



Here O is the initial point and A is the terminal point of OA

ADDITION OF TWO VECTORS



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{OB} = \vec{c}$.

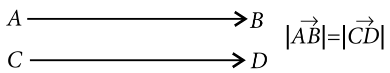
Here $\vec{c}$ is sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$. It is to be noticed that the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$ and the line joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ represents vector $\vec{a}+\vec{b}$ in magnitude and direction.

PROPERTIES

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$, (Vector addition is commutative)
- $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$, (Vector addition is associative)
- $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$, equality holds when $\vec{a}$ and $\vec{b}$ are like vectors.
- $|\vec{a}+\vec{b}|\ge||\vec{a}|-|\vec{b}||$, equality holds when $\vec{a}$ and $\vec{b}$ are unlike vectors.
- $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

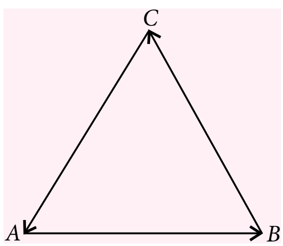
TYPES OF VECTORS

- **Equal vectors :** Two vectors are said to be equal if and only if they have equal magnitudes and same direction.



As well as their directions are also same

- **Zero Vector (null vector):** A vector whose initial and terminal points are same, is called the null vector. Such vector has zero magnitude and no direction, and denoted by $\vec{0}$.



$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AA} \quad \text{or} \quad \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$$

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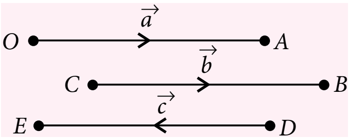
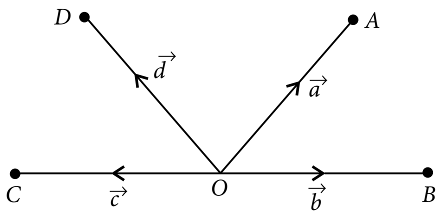
- **Like and Unlike Vectors:** Two vectors are said to be
  (i) Like, when they are in same direction.
  (ii) Unlike, when they are in opposite directions.

$\vec{a}$ and $-\vec{a}$ are two unlike vectors as their directions are opposite, $\vec{a}$ and $3\vec{a}$ are like vectors.

- **Unit Vector :** A unit vector is a vector whose magnitude is unity. We write, unit vector in the direction of $\vec{a}$ as $\hat{a}$. Therefore $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.
- **Parallel vectors:** Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vectors may have equal or unequal magnitudes and direction may be same or opposite.



- **Position Vector:** If P is any point in the space, then the vector $\overrightarrow{OP}$ is called position vector of point P, where O is the origin of reference. Thus for any two points A and B in the space, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$.
- **Co-initial vectors:** Vectors having same initial point are called co-initial vectors.



Here $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ are co-initial vectors.

MULTIPLICATION OF VECTOR BY SCALARS

If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called scalar multiplication. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then :

- $m(\vec{a})=(\vec{a})m=m\vec{a}$
- $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$
- $(m+n)\vec{a}=m\vec{a}+n\vec{a}$
- $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

LINEAR COMBINATION

Given a finite set of vectors $\vec{a},\vec{b},\vec{c},......$ then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+........$ is called a linear combination of $\vec{a},\vec{b},\vec{c},......$ for any $x, y, z .....\in R$. We have the following results:

- If $\vec{a},\vec{b}$ are non zero, non-collinear vectors then
  $x\vec{a}+y\vec{b}=x'\vec{a}+y'\vec{b} \Rightarrow x=x'; y=y'$
- **Fundamental Theorem :** Let $\vec{a},\vec{b}$ be non zero, non collinear vectors . Then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a},\vec{b}$ *i.e.* there exist some unique $x,y\in R$ such that $x\vec{a}+y\vec{b}=\vec{r}$.
- If $\vec{a},\vec{b},\vec{c}$ are non-zero, non-coplanar vectors then
  $x\vec{a}+y\vec{b}+z\vec{c}=x'\vec{a}+y'\vec{b}+z'\vec{c}$
  $\Rightarrow \quad x=x', y=y', z=z'$
- **Fundamental Theorem in Space :** Let $\vec{a},\vec{b},\vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ *i.e.* there exist some unique $x,y,z\in R$ such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.
- If $\vec{x}_1,\vec{x}_2,......,\vec{x}_n$ are n non-zero vectors and $k_1, k_2, ..., k_n$ are n scalars and if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+......+k_n\vec{x}_n=0$ $\Leftrightarrow k_1=0, k_2=0.....k_n=0$, then we say that vectors $\vec{x}_1,\vec{x}_2,......,\vec{x}_n$ are linearly independent vectors.
- If $\vec{x}_1,\vec{x}_2,......,\vec{x}_n$ are not linearly independent, then they are said to be linearly dependent vectors, *i.e.*, if $k_1\vec{x}_1+k_2\vec{x}_2+........+k_n\vec{x}_n=0$ and if there exists at least one $k_r \neq 0$, $(r=1, 2, ...n)$ then $\vec{x}_1,\vec{x}_2,......,\vec{x}_n$ are said to be linearly dependent.

LINEARLY DEPENDENT VECTOR

If $k_r \neq 0$;

$$k_1\vec{x}_1+k_2\vec{x}_2+k_3\vec{x}_3+.......+k_r\vec{x}_r+......+k_n\vec{x}_n=0$$

$$\Rightarrow -k_r\vec{x}_r=k_1\vec{x}_1+k_2\vec{x}_2+...+k_{r-1}\cdot\vec{x}_{r-1}+ k_{r+1}\cdot\vec{x}_{r+1}+...+k_n\vec{x}_n$$

$$\Rightarrow -k_r\frac{1}{k_r}\vec{x}_r=k_1\frac{1}{k_r}\vec{x}_1+k_2\frac{1}{k_r}\vec{x}_2+...+ k_{r-1}\cdot\frac{1}{k_r}\vec{x}_{r-1}+...+k_n\frac{1}{k_r}\vec{x}_n$$

$$\Rightarrow \vec{x}_r=c_1\vec{x}_1+c_2\vec{x}_2+....+c_{r-1}\vec{x}_{r-1}+c_r\vec{x}_{r+1}+.... +c_{n-1}\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1, \vec{x}_2,....,\vec{x}_{r-1},\vec{x}_{r+1},....,\vec{x}_n$

Hence, $\vec{x}_r$ with $\vec{x}_1,\vec{x}_2,....,\vec{x}_{r-1},\vec{x}_{r+1},....,\vec{x}_n$ forms a linearly dependent set of vectors .

- If $\vec{a}=3\hat{i}+2\hat{j}+5\hat{k}$, then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i},\hat{j},\hat{k}$. Also, $\vec{a}$, $\hat{i},\hat{j},\hat{k}$ form a linearly dependent set of vectors. In general , every set of four vectors is a linearly dependent system.

- $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = K_2 = K_3 = 0$.

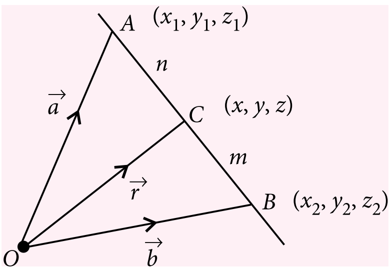
COLLINEARITY AND COPLANARITY OF POINTS

The necessary and sufficient condition for three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be collinear is that there exist scalars x, y, z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$

The necessary and sufficient condition for four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ to be coplanar is that there exist scalars x, y, z ,u (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$ where $x + y + z + u = 0$.

SECTION FORMULA

Let A, B and C be three collinear points in space having position vectors $\vec{a}, \vec{b}$ and $\vec{r}$ respectively.



Let $\dfrac{AC}{CB} = \dfrac{n}{m}$

or, $m\,AC = n\,CB$ or, $m\overline{AC} = n\overline{CB}$ ...(i)

(As vectors are in same direction)

Now, $\overline{OA} + \overline{AC} = \overline{OC} \Rightarrow \overline{AC} = \vec{r} - \vec{a}$ ...(ii)

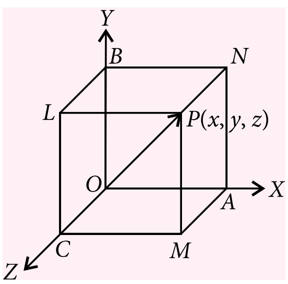
$\vec{r} + \overline{CB} = \vec{b} \Rightarrow \overline{CB} = \vec{b} - \vec{r}$ ...(iii)

Using (i), we get $\vec{r} = \dfrac{m\vec{a} + n\vec{b}}{m+n}$

ORTHOGONAL SYSTEM OF UNIT VECTORS

Let OX, OY and OZ be three mutually perpendicular straight lines. Given any point $P(x, y, z)$ in space, we can construct the rectangular parallelopiped of which OP is a diagonal and $OA = x$, $OB = y$, $OC = z$.

Here A, B, C are $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ respectively and L, M, N are $(0, y, z)$,$(x, 0, z)$ and $(x, y, 0)$ respectively.



Let $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along OX, OY and OZ respectively.

We have $\vec{r} = \overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ as $\overline{OA} = x\hat{i}$, $\overline{OB} = y\hat{j}$ and $\overline{OC} = z\hat{k}$

$\overline{ON} = \overline{OA} + \overline{AN}$; $\overline{OP} = \overline{ON} + \overline{NP}$

So, $\overline{OP} = \overline{OA} + \overline{OB} + \overline{OC}$ $\left(\because\ \overline{NP} = \overline{OC},\ \overline{AN} = \overline{OB}\right)$

$$r = |\vec{r}| = |\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow\ \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$\Rightarrow\ \hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$$

SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT)

The scalar product, $\vec{a} \cdot \vec{b}$ of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as $|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors, when drawn with same initial point.

Note that $0 \le \theta \le \pi$.

If at least one of $\vec{a}$ and $\vec{b}$ is a zero vector, then $\vec{a} \cdot \vec{b}$ is defined as zero.

PROPERTIES

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (scalar product is commutative)
- $\vec{a}^2 = \vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$ (where m is a scalar)
- $\theta = \cos^{-1}\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow$ Vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other. [provided $\vec{a}, \vec{b}$ are non-zero vectors]
- $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$
- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

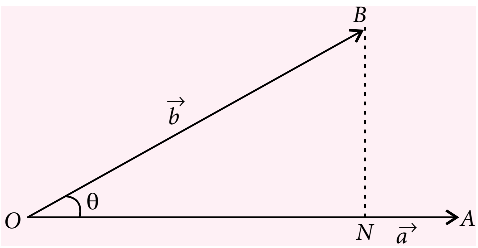
  Then $\vec{a} \cdot \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

  $= a_1b_1 + a_2b_2 + a_3b_3$
- Maximum value of $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
- Minimum value of $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

- Any vector $\vec{a}$ can be written as,
  $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

Algebraic projection of a vector along some other vector :

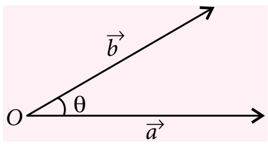
$$ON = OB\cos\theta = |\vec{b}|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \hat{a}\cdot\vec{b}$$



VECTOR (CROSS) PRODUCT

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, whose modulus are a and b respectively, is the vector whose modulus is $ab\sin\theta$, where $\theta(0 \le \theta \le \pi)$is the angle between the vectors $\vec{a}$ and $\vec{b}$. Its direction is that of a vector $\vec{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \vec{n}$ are in right-handed orientation.

By the right-handed orientation we mean that, if we turn the vector $\vec{a}$ into the vector $\vec{b}$ through the angle θ, then $\vec{n}$ points in the direction in which a right handed screw would move if turned in the same manner. Thus $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$.



If at least one of $\vec{a}$ and $\vec{b}$ is a zero vector, then $\vec{a}\times\vec{b}$ is defined as the zero vector.

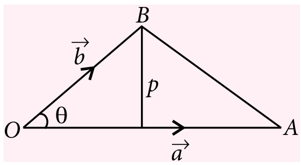
PROPERTIES

- $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$
- $(m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$ (where m is a scalar)
- $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow$ vectors $\vec{a}$ and $\vec{b}$ are parallel.
  (provided $\vec{a}$ and $\vec{b}$ are non-zero vectors)
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- $\hat{i}\times\hat{j} = \hat{k} = -(\hat{j}\times\hat{i}), \hat{j}\times\hat{k} = \hat{i} = -(\hat{k}\times\hat{j})$,
  $\hat{k}\times\hat{i} = \hat{j} = -(\hat{i}\times\hat{k})$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

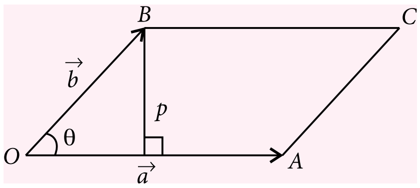
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

- $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$
- Area of $\Delta OAB = \dfrac{1}{2}|\vec{a}\times\vec{b}| = \dfrac{1}{2}ab\sin\theta$



- Area of parallelogram $OACB = ab\sin\theta = |\vec{a}\times\vec{b}|$.



- $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ (not commutative)
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
- A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.
- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$.
- **Lagrange's Identity :** For any two vectors $\vec{a}$ and $\vec{b}$;
  $$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

SCALAR TRIPLE PRODUCT (BOX PRODUCT)

The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $(\vec{a}\times\vec{b})\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and ϕ is the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$. It is also defined as $[\vec{a}\,\vec{b}\,\vec{c}]$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

$$=\hat{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\hat{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\hat{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=c_1\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-c_2\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+c_3\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Therefore, $(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b}$

$$=-(\vec{b}\times\vec{a})\cdot\vec{c}=-(\vec{c}\times\vec{b})\cdot\vec{a}=-(\vec{a}\times\vec{c})\cdot\vec{b}$$

Note that $(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=\vec{a}\cdot(\vec{b}\times\vec{c})$, hence in scalar triple product, dot and cross are interchangeable. Therefore, we denote $(\vec{a}\times\vec{b})\cdot\vec{c}$ by $[\vec{a}\ \vec{b}\ \vec{c}]$.

PROPERTIES

- $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ represents the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{a},\vec{b}$ and $\vec{c}$ in magnitude and direction. Therefore three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

  i.e., $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0$

- In a scalar triple product the position of dot and cross can be interchanged *i.e.* $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ *i.e.*, $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$ *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$
- In general, if $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ and $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$

  then $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\left[\vec{l}\ \vec{m}\ \vec{n}\right]$;

  where $\vec{l},\vec{m}$ and $\vec{n}$ are non-coplanar vectors.
- Scalar product of three vectors, two of which are equal or parallel is 0.

Note : If $\vec{a},\vec{b},\vec{c}$ are non - coplanar then $[\vec{a}\ \vec{b}\ \vec{c}]>0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}]<0$ for left handed system.

- $[\hat{i}\ \hat{j}\ \hat{k}]=1$.
- $[K\vec{a}\ \vec{b}\ \vec{c}]=K[\vec{a}\ \vec{b}\ \vec{c}]$.
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}]=[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$
- The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are given by $\frac{1}{4}\left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)$.

Note that this is also the point of concurrence of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Remember that : $\left[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}\right]=0$

and $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is defined as $\vec{a}\times(\vec{b}\times\vec{c})$. If at least one $\vec{a},\vec{b}$ and $\vec{c}$ is a zero vector or $\vec{b}$ and $\vec{c}$ are collinear vectors or $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, only then $\vec{a}\times(\vec{b}\times\vec{c})=\vec{0}$. In all other cases $\vec{a}\times(\vec{b}\times\vec{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector $\vec{a}$.

Thus we can take $\vec{a}\times(\vec{b}\times\vec{c})=\lambda\vec{b}+\mu\vec{c}$, for some scalars λ and μ.

Since, $\vec{a}\perp\vec{a}\times(\vec{b}\times\vec{c})\Rightarrow a\cdot(\vec{a}\times(\vec{b}\times\vec{c}))=0$

$\Rightarrow \lambda(\vec{a}\cdot\vec{b})+\mu(\vec{a}\cdot\vec{c})=0$

$\Rightarrow \lambda=(\vec{a}\cdot\vec{c})\alpha,\ \mu=-(\vec{a}\cdot\vec{b})\alpha$, for some scalar α.

Hence, $\vec{a}\times(\vec{b}\times\vec{c})=\alpha[(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c})]$, for any vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfying the conditions given in the beginning. In particular if we take, $\vec{a}=\vec{b}=\hat{i},\ \vec{c}=\hat{j}$, then $\alpha=1$.

Hence, $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a},\vec{b}$ and $\vec{c}$ be a system of three non-coplanar vectors. Then the system of vectors $\vec{a}',\vec{b}'$ and $\vec{c}'$ which satisfies $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$ and $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=\vec{b}\cdot\vec{a}'=\vec{b}\cdot\vec{c}'=\vec{c}\cdot\vec{a}'=\vec{c}\cdot\vec{b}'=0$, is called the reciprocal system to

Rvp

the vectors $\vec{a}, \vec{b}, \vec{c}$. In terms of $\vec{a}, \vec{b}, \vec{c}$ the vectors $\vec{a}', \vec{b}', \vec{c}'$ are given by $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$.

PROPERTIES

- $\vec{a} \cdot \vec{b}' = \vec{a} \cdot \vec{c}' = \vec{b} \cdot \vec{a}' = \vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = \vec{c} \cdot \vec{b}' = 0$
- The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ formed from three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is the reciprocal of the scalar triple product formed from reciprocal system.

SOLVING OF VECTOR EQUATION

Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions)

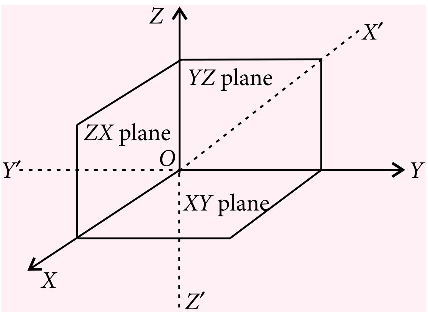
Generally, to solve vector equations, we express the unknown as the linear combination of three non-coplanar vectors as

$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a} \times \vec{b})$ as $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are non-coplanar and find x, y, z using given conditions. Sometimes we can directly solve the given conditions it would be more clear from some examples.

THREE - DIMENSIONAL GEOMETRY

RECTANGULAR COORDINATE SYSTEM IN SPACE

Let 'O' be any point in space and three lines are perpendicular to each other. These lines are known as coordinate axes and O is called origin. The planes XY, YZ, ZX are known as the coordinate planes.



COORDINATES OF A POINT IN SPACE

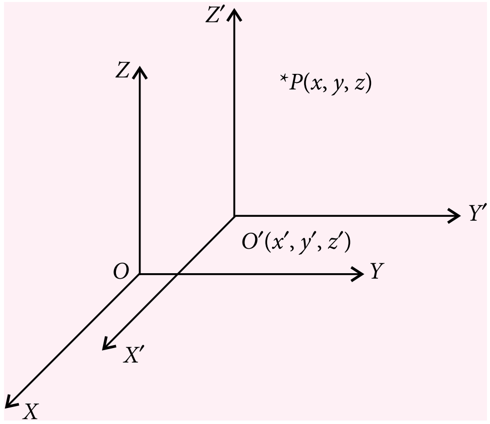
Consider a point P in space whose position is given by triad (x, y, z) where x, y, z are perpendicular distance from YZ-plane, ZX-plane and XY-plane respectively.

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along OX, OY, OZ respectively, then position vector of point P is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (x, y, z).

SHIFTING OF ORIGIN

Shifting the origin to another point without changing the directions of the axes is called the translation of axes.

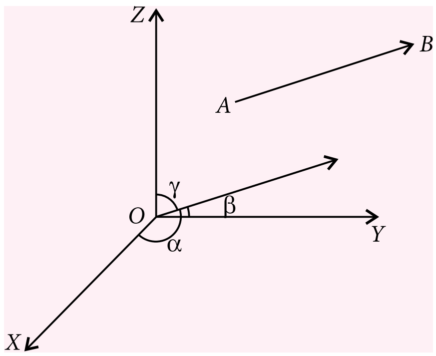
Let the origin O be shifted to another point without changing the direction of axes.



Let the new coordinate frame be $O'\ X'\ Y'\ Z'$. Let $P(x, y, z)$ be a point with respect to the coordinate frame $OXYZ$. Then, coordinate of point P w.r.t. new coordinate frame $O'\ X'\ Y'\ Z'$ is (x_1, y_1, z_1), where, $x_1 = x - x'$, $y_1 = y - y'$, $z_1 = z - z'$ where O is shifted at $O'(x', y', z')$

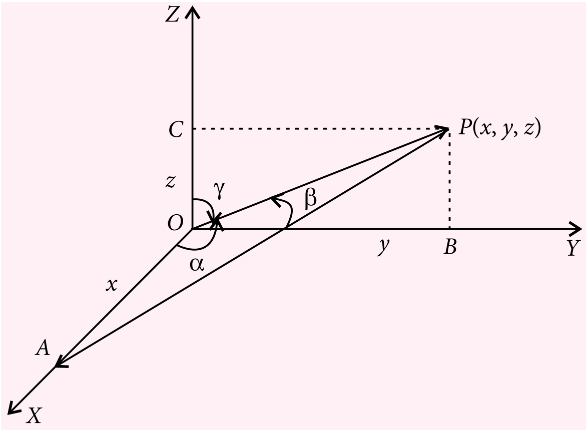
DIRECTION COSINES & DIRECTION RATIOS OF A LINE

If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z coordinate axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the given line and are generally denoted by l, m, n respectively. Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. By definition it follows that the direction cosine of the x-axis is cos 0°, cos 90°, cos 90° *i.e.* (1, 0, 0). Similarly direction cosines of the y-axis and z-axis are respectively (0, 1, 0) and (0, 0, 1).



Note :

- The unit vector along the line can be written as $l\hat{i}+m\hat{j}+n\hat{k}$
- If d . c's of line AB is (l, m, n), then d . c's of line BA will be $(-l, -m, -n)$
- Let OP be any line through the origin O which has direction cosines l, m, n. Let $P(x, y, z)$ and $OP = r$.



Then $OP^2 = x^2 + y^2 + z^2 = r^2$ ...(1)

From P draw PA, PB, PC perpendicular on the coordinate axes, so that $OA = x$, $OB = y$, $OC = z$

Also, $\angle POA = \alpha$, $\angle POB = \beta$ and $\angle POC = \gamma$.

From ΔAOP, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly, $y = mr$ and $z = nr$

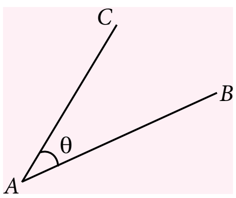
Hence from (1)

$r^2(l^2 + m^2 + n^2) = r^2 \quad \Rightarrow \quad l^2 + m^2 + n^2 = 1$

- If the coordinates of any point P be (x, y, z) and l, m, n be the direction cosines of the line OP, O being the origin, then (lr, mr, nr) will give us the coordinates of a point on the line OP which is at a distance r from $(0, 0, 0)$.

ANGLE BETWEEN TWO LINES

Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.



If direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by

by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Result : We have

$$\sin^2\theta = 1 - \cos^2\theta = (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$$

$$\Rightarrow \sin\theta = \pm\sqrt{\sum(l_1m_2 - l_2m_1)^2}$$

CONDITION OF PERPENDICULARITY

If the given lines are perpendicular, then $\theta = 90°$ *i.e.* $\cos\theta = 0$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

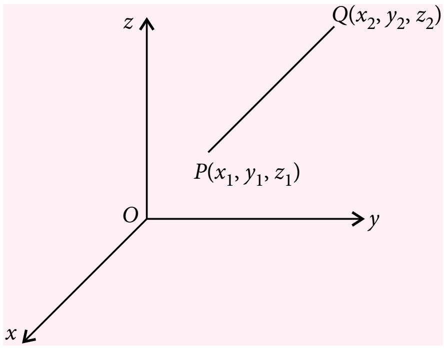
CONDITION OF PARALLELISM

If the given lines are parallel, then $\theta = 0°$ *i.e.* $\sin\theta = 0$

$$\Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

DIRECTION COSINES OF A LINE PASSING THROUGH TWO POINTS

The direction ratios of line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$(say), $y_2 - y_1 = b$(say) and $z_2 - z_1 = c$(say). Then direction cosines are



$$l = \pm\frac{(x_2 - x_1)}{\sqrt{\sum(x_2 - x_1)^2}}, \quad m = \pm\frac{(y_2 - y_1)}{\sqrt{\sum(y_2 - y_1)^2}},$$

$$n = \pm\frac{(z_2 - z_1)}{\sqrt{\sum(z_2 - z_1)^2}}$$

THE STRAIGHT LINE

Straight line in three dimensional geometry is defined as intersection of two planes. So general equation of straight line is stated as the equations of both planes together *i.e.* general equation of straight line is

$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$ ....(1)

So, equation (1) represents straight line which is obtained by intersection of two planes.

EQUATION OF STRAIGHT LINE IN DIFFERENT FORMS

Equation of straight line passing through point $P(x_1, y_1, z_1)$ and whose direction cosines are l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.$$

Vector equation of straight line is $\vec{r}=\vec{r}_1+\lambda\vec{b}$ where $\vec{r}_1$ is the position vector of a point in the straight line and $\vec{b}$ is a vector parallel to the straight line.

Equation of straight line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

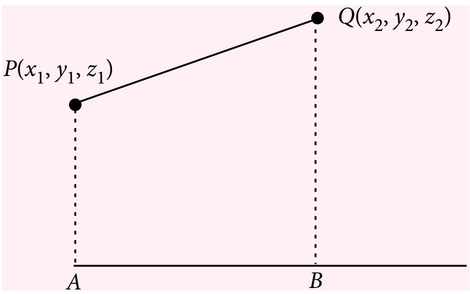
Vector form : Equation of straight line passing through two points $P(\vec{r}_1)$ and $Q(\vec{r}_2)$ is $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$

Note : The general coordinates of a point on a line is given by $(x_1 + lr, y_1 + mr, z_1 + nr)$ where r is distance between the point (x_1, y_1, z_1) and point whose coordinates is to be written.

PROJECTION OF A LINE SEGMENT ON A GIVEN LINE

Projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n is $AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.

Here PQ and AB need not be coplanar.



COPLANARITY OF TWO LINES

In vector form

If the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ are coplanar, then

$$\left[\vec{a}_1\,\vec{b}_1\,\vec{b}_2\right]=\left[\vec{a}_2\,\vec{b}_1\,\vec{b}_2\right]$$

and the equation of plane containing them is

$$\left[\vec{r}\,\vec{b}_1\,\vec{b}_2\right]=\left[\vec{a}_1\,\vec{b}_1\,\vec{b}_2\right]\text{or}\left[\vec{r}\,\vec{b}_1\,\vec{b}_2\right]=\left[\vec{a}_2\,\vec{b}_1\,\vec{b}_2\right]$$

In cartesian form, if the lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}\text{ and }\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

then condition of coplanarity is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

and the equation of plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

SHORTEST DISTANCE BETWEEN TWO SKEW LINES (NON-PARALLEL, NON-INTERSECTING LINES)

Two non-parallel, non-intersecting lines are called skew lines. The shortest distance between these two lines is the distance of the intercepting portion of a line perpendicular to both the lines.

Method : Let the equation of two non-intersecting lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}=r_1(\text{say}) \quad \text{...(1)}$$

$$\text{and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2(\text{say}) \quad \text{...(2)}$$

Any point on line (1) is $P(x_1 + l_1r_1, y_1 + m_1r_1, z_1 + n_1r_1)$ and on line (2) is $Q(x_2 + l_2r_2, y_2 + m_2r_2, z_2 + n_2r_2)$.

Let PQ be the line of shortest distance, its direction ratios will be

$[(l_1r_1 + x_1 - x_2 - l_2r_2), (m_1r_1 + y_1 - y_2 - m_2r_2), (n_1r_1 + z_1 - z_2 - n_2r_2)]$

This line is perpendicular to both the given lines. By using condition of perpendicularity we obtain two equations in r_1 and r_2.

So by solving these, values of r_1 and r_2 and subsequently point P and Q can be found. The distance PQ is the shortest distance.

Note : The shortest distance between two lines can be given by

$$\frac{1}{\sqrt{\sum(l_1m_2-l_2m_1)^2}}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

The shortest distance between two lines

$$\vec{r}=\vec{a}_1+\lambda\vec{b}_1 \text{ and } \vec{r}=\vec{a}_2+\mu\vec{b}_2 \text{ is } \left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

If any straight line is given in general form then it can be transformed into symmetrical form and we can further proceed.

SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$, where $\vec{a}_1$ is the position vector of a point on l_1 and $\vec{a}_2$ is the position vector of a point on l_2.

The shortest distance between the given parallel lines is $d = \left|\dfrac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

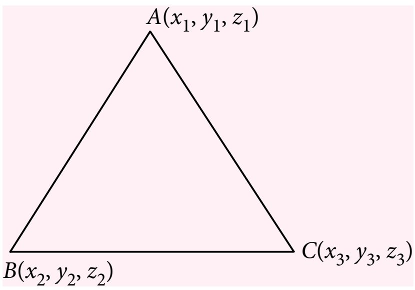
EQUATION OF THE PLANE THROUGH A GIVEN LINE

- If equation of line is given in general form as $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ then the equation of plane passing through this line is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$.
- If equation of the line is given in symmetrical form as $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$, then equation of plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are given by $al + bm + cn = 0$.

AREA OF A TRIANGLE

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix},$$

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



So, area of ΔABC is given by the relation

$$\Delta^2 = \Delta_x{}^2 + \Delta_y{}^2 + \Delta_z{}^2$$

VOLUME OF TETRAHEDRON

Let $\vec{a}, \vec{b}, \vec{c}$ be 3 coterminous edges of a tetrahedron.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Volume of tetrahedron $= \dfrac{1}{6}\times$ volume of parallelopiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$

$$= \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

THE PLANE

Consider the locus of a point $P(x, y, z)$. If x, y, z are allowed to vary with any restriction for their different combinations, we have a set of points for P. The surface on which these points lie, is called the locus of P. It may be a plane or any curved surface. If Q be any other point on its locus and all points of the straight line PQ lie on it then it is a plane. In other words for the plane the straight line PQ, however small and in whatever direction will completely lie on the locus, otherwise it will be a curved surface.

EQUATION OF A PLANE IN NORMAL FORM, PASSING THROUGH A FIXED POINT

General equation of a plane is $ax + by + cz + d = 0$. (where a, b, c gives the direction ratios of the normal to the plane).

Equation of the plane in normal form is $lx + my + nz = p$ where p is the length of the normal from the origin to the plane and (l, m, n) be the direction cosines of the normal .

Vector form : Equation of the plane in vector form, $\vec{r}\cdot\hat{n} = d$, where $\hat{n}$ is a unit vector perpendicular to the plane and d is length of perpendicular drawn from origin to the plane.

Equation of YZ plane is $x = 0$,
equation of plane parallel to YZ plane is $x = d$.
Equation of ZX plane is $y = 0$,
equation of plane parallel to ZX plane is $y = d$.
Equation of XY plane is $z = 0$,
equation of plane parallel to XY plane is $z = d$.

Four points namely $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ will be coplanar if one point lies on the plane passing through other three points.

EQUATION OF A PLANE PERPENDICULAR TO A GIVEN LINE AND PASSING THROUGH A GIVEN POINT

The equation to the plane passing through $P(x_1, y_1, z_1)$ and is perpendicular to given line having direction ratios (a, b, c) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Vector form : The equation to the plane passing through a point whose position vector is $\vec{r}_1$ and unit vector perpendicular to the plane is $\hat{n}$, is $(\vec{r}-\vec{r}_1)\cdot\hat{n}=0$

EQUATION OF A PLANE PASSING THROUGH THREE NON COLLINEAR POINTS

The equation of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is
$$\begin{vmatrix} (x-x_1) & (y-y_1) & (z-z_1) \\ (x_2-x_1) & (y_2-y_1) & (z_2-z_1) \\ (x_3-x_1) & (y_3-y_1) & (z_3-z_1) \end{vmatrix}=0.$$

INTERCEPT FORM OF THE EQUATION OF A PLANE

Intercept form of the equation of a plane is given by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a, b, c are x-intercept, y-intercept, z-intercept respectively .

DISTANCE OF A POINT FROM A PLANE

The length of the perpendicular of the point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Vector form : The length of the perpendicular from the point $P(\vec{r}_1)$ to the plane $\vec{r}.\hat{n}=d$ is $|\vec{r}_1.\hat{n}-d|$

Note :

The distance between two parallel planes is the algebraic difference of perpendicular distances on the planes from origin.

Let two parallel planes be $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$

Then distance between them is given by $\frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$

ANGLE BETWEEN A LINE AND A PLANE

If equation of a plane is $ax + by + cz + d = 0$, then direction ratios of normal to this plane are a, b, c.

The equation of straight line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, then angle between normal to the plane and straight line is given by $\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$, where θ is the angle between the plane and the line.

The plane and the straight line will be parallel if $al + bm + cn = 0$

The plane and the straight line will be perpendicular if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

POSITION OF TWO POINTS W.R.T. A PLANE

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q are lying on same or opposite sides of the plane.

ANGLE BETWEEN TWO PLANES

Angle between the planes is defined as angle between normals to the planes drawn from any point. Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is
$$\cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a^2{}_1+b^2{}_1+c^2{}_1\right)\left(a^2{}_2+b^2{}_2+c^2{}_2\right)}}\right).$$

Vector form : Angle between the planes is
$$\cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$
where $\vec{n}_1$ and $\vec{n}_2$ are vectors perpendicular to the planes.

Note : If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ then the planes are parallel to each other.

PLANE PASSING THROUGH THE INTERSECTION OF TWO GIVEN PLANES

Equation of plane passing through the line of intersection of two planes $u = 0$ and $v = 0$ is $u + \lambda v = 0$.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. The length of the perpendicular from the origin to the plane passing through the point $\vec{a}$ & containing the line $\vec{r}=\vec{b}+\lambda\vec{c}$ is

(a) $\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$ (b) $\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}|}$

(c) $\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$ (d) $\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}$

2. Equation of the plane through (3, 4, –1) which is parallel to the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+7=0$ is

(a) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$

(b) $\vec{r}\cdot(3\hat{i}+4\hat{j}-\hat{k})+11=0$

(c) $\vec{r}\cdot(3\hat{i}+4\hat{j}-\hat{k})+7=0$

(d) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})-7=0$

3. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}=x\hat{i}+(x-2)\hat{j}-\hat{k}$. If the vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then x equals

(a) 0 (b) 1 (c) –4 (d) –2

4. Equation of the plane containing the lines

$\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}+2\hat{j}-\hat{k})$ and

$\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\mu(\hat{i}+\hat{j}+3\hat{k})$ is

(a) $\vec{r}\cdot(7\hat{i}-4\hat{j}-\hat{k})=0$

(b) $7(x-1)-4(y-1)-(z+3)=0$

(c) $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=0$

(d) $\vec{r}\cdot(\hat{i}+\hat{j}+3\hat{k})=0$

5. The cartesian equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$ is

(a) $3x-4y+4z=5$ (b) $x-2y+4z=3$

(c) $5x-2y-12z+47=0$ (d) $2x+3y+4=0$

6. If $\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})=\vec{0}$, then

(a) $\vec{a},\vec{b},\vec{c}$ are coplanar only if none of α, β, γ is zero

(b) $\vec{a},\vec{b},\vec{c}$ are coplanar if atleast one of α, β, γ is non zero

(c) $\vec{a},\vec{b},\vec{c}$ are non-coplanar for any α, β, γ

(d) none of these

7. If $((\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}))\cdot(\vec{a}\times\vec{d})=0$, then which of the following is always true

(a) $\vec{a},\vec{b},\vec{c},\vec{d}$ are necessarily coplanar

(b) either $\vec{a}$ or $\vec{d}$ must lie in the plane of $\vec{b}$ or $\vec{c}$

(c) either $\vec{b}$ or $\vec{c}$ must lie in plane of $\vec{a}$ and $\vec{d}$

(d) either $\vec{a}$ or $\vec{b}$ must lie in plane of $\vec{c}$ and $\vec{d}$

8. Let $\vec{r}=(\vec{a}\times\vec{b})\sin x+(\vec{b}\times\vec{c})\cos y+2(\vec{c}\times\vec{a})$, where $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors. If $\vec{r}$ is perpendicular to $\vec{a}+\vec{b}+\vec{c}$, then minimum value of x^2+y^2 is

(a) π^2 (b) $\dfrac{\pi^2}{4}$

(c) $\dfrac{5\pi^2}{4}$ (d) none of these

9. If the two adjacent sides of two rectangles are represented by the vectors $\vec{p}=5\vec{a}-3\vec{b};\ \vec{q}=-\vec{a}-2\vec{b}$ and $\vec{r}=-4\vec{a}-\vec{b};\vec{s}=-\vec{a}+\vec{b}$ respectively, then the angle between the vectors $\vec{x}=\frac{1}{3}(\vec{p}+\vec{r}+\vec{s})$ and $\vec{y}=\frac{1}{5}(\vec{r}+\vec{s})$ is

(a) $-\cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$ (b) $\cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$

(c) $\pi-\cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$ (d) cannot be evaluated

10. The lengths of two opposite edges of a tetrahedron are a, b. Their shortest distance is d and the angle between them is θ. Then its volume is

(a) $\dfrac{1}{2}abd\sin\theta$ (b) $\dfrac{1}{3}abd\cos\theta$

(c) $\dfrac{1}{6}abd\sin\theta$ (d) $\dfrac{1}{6}abd\cos\theta$

11. The position vectors of the points A and B with respect to a fixed point O (origin) are $2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}+4\hat{k}$. The length of the internal bisector of $\angle BOA$ of ΔAOB is

(a) $\dfrac{\sqrt{136}}{9}$ (b) $\dfrac{\sqrt{136}}{3}$ (c) $\dfrac{20}{3}$ (d) $\dfrac{\sqrt{217}}{9}$

12. Equation of a plane bisecting the angle between the planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$ is

(a) $5x-y-4z-45=0$

(b) $5x-y-4z-3=0$

(c) $23x+13y+32z-45=0$

(d) $23x-13y+32z+5=0$

13. If the perpendicular distance of a point P other than the origin from the plane $x+y+z=p$ is equal to the distance of the plane from the origin, then the coordinates of P are

(a) $(p, 2p, 0)$ (b) $(0, 2p, -p)$

(c) $(2p, p, -p)$ (d) $(2p, -p, 2p)$

14. If the angle θ between the line $\dfrac{x+1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{2}$ and the plane $2x-y+\sqrt{\lambda}z+4=0$ is such that $\sin\theta=1/3$, then the value of λ is

(a) 3/4 (b) –4/3 (c) 5/3 (d) –3/5

15. If l_1, m_1, n_1 and l_2, m_2, n_2, are d.c.'s of $\overline{OA}, \overline{OB}$ such that $\angle AOB = \theta$, where 'O' is the origin, then the d.c.'s of the internal bisector of the angle $\angle AOB$ are

(a) $\frac{l_1+l_2}{2\sin\theta/2}, \frac{m_1+m_2}{2\sin\theta/2}, \frac{n_1+n_2}{2\sin\theta/2}$

(b) $\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}$

(c) $\frac{l_1-l_2}{2\sin\theta/2}, \frac{m_1-m_2}{2\sin\theta/2}, \frac{n_1-n_2}{2\sin\theta/2}$

(d) $\frac{l_1-l_2}{2\cos\theta/2}, \frac{m_1-m_2}{2\cos\theta/2}, \frac{n_1-n_2}{2\cos\theta/2}$

16. If a line with direction ratios 2 : 2 : 1 intersects the line $\frac{x-7}{3}=\frac{y-5}{2}=\frac{z-3}{1}$ and $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z+1}{3}$ at A and B respectively, then $AB =$

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 3

17. The plane $x - y - z = 4$ is rotated through 90° about its line of intersection with the plane $x + y + 2z = 4$ and equation in new position is $Ax + By + Cz + D = 0$ where A,B,C are least positive integers and $D < 0$ then

(a) $D = -10$ (b) $ABC = -20$
(c) $A + B + C + D = 0$ (d) $A + B + C = 10$

18. The reflection of the point $P(1, 0, 0)$ in the line $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ is

(a) $(3, -4, -2)$ (b) $(5, -8, -4)$
(c) $(1, -1, -10)$ (d) $(2, -3, 8)$

19. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

(a) $x + 2y - 2z = 0$ (b) $3x + 2y - 2z = 0$
(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

20. The two lines whose direction cosines are connected by the relations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are perpendicular, then

(a) $a^2(v - w) + b^2(w - u) + c^2(u - v) = 0$

(b) $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$

(c) $a(v^2 + w^2) + b(w^2 + u^2) + c(u^2 + v^2) = 0$

(d) $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$

21. Through a point $P(h, k, l)$ a plane is drawn at right angles to OP to meet the co-ordinate axes in A, B and C. If $OP = p$, then the area of ΔABC is

(a) $\frac{p^2hk}{l^2}$ (b) $\frac{p^3l}{3hk}$ (c) $\frac{p^2l^2}{2hk}$ (d) $\frac{p^5}{2hkl}$

22. If $l_i^2+m_i^2+n_i^2=1\ \forall\ i\in\{1,2,3\}$ and $l_il_j+m_im_j+n_in_j=0\ \forall\ i,j\in\{1,2,3\}\ (i\neq j)$,

$$\Delta=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \text{ then}$$

(a) $|\Delta| = 3$ (b) $|\Delta| = 2$ (c) $|\Delta| = 1$ (d) $\Delta = 0$

SECTION-II

Multiple Correct Answer Type

23. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|=$

(a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$
(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (d) $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

24. If $\vec{a}$ and $\vec{b}$ are two unit vectors perpendicular to each other and $\vec{c}=\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3(\vec{a}\times\vec{b})$, then the following is (are) true

(a) $\lambda_1=\vec{a}\cdot\vec{c}$ (b) $\lambda_2=|\vec{b}\times\vec{a}|$
(c) $\lambda_3=|(\vec{a}\times\vec{b})\times\vec{c}|$
(d) $\lambda_1+\lambda_2+\lambda_3=(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot\vec{c}$

25. If a vector $\vec{r}$ satisfies the equation $\vec{r}\times(\hat{i}+2\hat{j}+\hat{k})=\hat{i}-\hat{k}$, then $\vec{r}$ is equal to

(a) $\hat{i}+3\hat{j}+\hat{k}$ (b) $3\hat{i}+7\hat{j}+3\hat{k}$
(c) $\hat{j}+t(\hat{i}+2\hat{j}+\hat{k})$, where t is any scalar
(d) $\hat{i}+(t+3)\hat{j}+\hat{k}$, where t is any scalar

26. In a four-dimensional space, where unit vectors along axes are $\hat{i},\hat{j},\hat{k}$ and $\hat{l}$ and $\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda-1)(\vec{a}_1-\vec{a}_2)+\mu(\vec{a}_2+\vec{a}_3)+\gamma(\vec{a}_3+\vec{a}_4-2\vec{a}_2)+\vec{a}_3+\delta\vec{a}_4=\vec{0}$, then

(a) $\lambda = 1$ (b) $\mu=-\frac{2}{3}$
(c) $\lambda=\frac{2}{3}$ (d) $\delta=\frac{1}{3}$

27. Identify the statement(s) which is/are incorrect ?

(a) $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]=(\vec{a}\times\vec{b})(\vec{a}^2)$
(b) If $\vec{a},\vec{b},\vec{c}$ are non-zero, non coplanar vector and $\vec{v}\cdot\vec{a}=\vec{v}\cdot\vec{b}=\vec{v}\cdot\vec{c}=0$, then $\vec{v}$ must be a null vector.

(c) If $\vec{a}$ and $\vec{b}$ lie in a plane normal to the plane containing the vectors $\vec{c}$ and $\vec{d}$, then $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$

(d) If $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are reciprocal system of vectors, then $\vec{a}\cdot\vec{b}'+\vec{b}\cdot\vec{c}'+\vec{c}\cdot\vec{a}'=3$

28. The position vector of a point P is $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, when $x, y, z \in N$ and $\vec{a}=\hat{i}+\hat{j}+\hat{k}$. If $\vec{r}\cdot\vec{a}=10$, the number of possible position of P is

(a) 36 (b) 72 (c) 66 (d) ${}^9C_2$

29. The projection of line $3x - y + 2z - 1 = 0 = x + 2y - z - 2$ on the plane $3x + 2y + z = 0$ is

(a) $\frac{x+1}{11}=\frac{y-1}{-9}=\frac{z-1}{-15}$

(b) $3x - 8y + 7z + 4 = 0 = 3x + 2y + z$

(c) $\frac{x+12}{11}=\frac{y+8}{-9}=\frac{z+14}{15}$

(d) $\frac{x+12}{11}=\frac{y+8}{-9}=\frac{z+14}{-15}$

30. The equation of three planes are $x - 2y + z = 3$, $5x - y - z = 8$ and $x + y - z = 7$, then

(a) they form a triangular prism

(b) all three plane have a common line of intersection

(c) line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is parallel to each plane

(d) line $\frac{x}{1}=\frac{y}{3}=\frac{z}{4}$ intersect all three plane

31. If the median through A of ΔABC having vertices $A \equiv (2, 3, 5)$, $B \equiv (-1, 3, 2)$ and $C \equiv (\lambda, 5, \mu)$ is equally inclined to the axes, then

(a) $\lambda = 7$ (b) $\mu = 10$ (c) $\lambda = 10$ (d) $\mu = 7$

32. Consider the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$. The plane $67x - 162y + 47z + 44 = 0$ bisects that angle between the given planes which

(a) contains origin (b) is acute

(c) is obtuse (d) none of these

33. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$, through an angle α, then equation of plane in its new position may be

(a) $lx+my+z\sqrt{l^2+m^2}\tan\alpha=0$

(b) $lx+my-z\sqrt{l^2+m^2}\tan\alpha=0$

(c) data is not sufficient

(d) none of these

SECTION-III

Comprehension Type

Paragraph for Question No. 34 to 35

Consider the equations of planes

$P_1\equiv\vec{r}\cdot(\hat{i}+2\hat{j}+\hat{k})-3=0,\ P_2\equiv\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})-5=0$

34. The equation of plane passing through the intersection of $P_1 = 0$, $P_2 = 0$ and through the point $A(1, 1, 1)$ is

(a) $\vec{r}\cdot(5\hat{i}-4\hat{j}+5\hat{k})=6$ (b) $\vec{r}\cdot(5\hat{i}+5\hat{j}-4\hat{k})=6$

(c) $\vec{r}\cdot(5\hat{i}+5\hat{j}+4\hat{k})=14$ (d) none of these

35. The line of intersection of planes $P_1 = 0$, $P_2 = 0$ is parallel to

(a) $3\hat{i}-5\hat{j}-\hat{k}$ (b) $3\hat{i}+\hat{j}-5\hat{k}$

(c) $2\hat{i}-\hat{j}-\hat{k}$ (d) none of these

Paragraph for Question No. 36 to 38

Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a}_1$ be the projection of $\vec{a}$ on $\vec{b}$ and $\vec{a}_2$ be the projection of $\vec{a}_1$ on $\vec{c}$. Then

36. $\vec{a}_2 =$

(a) $\frac{943}{49}(2\hat{i}-3\hat{j}-6\hat{k})$ (b) $\frac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$

(c) $\frac{943}{49}(-2\hat{i}+3\hat{j}+6\hat{k})$ (d) $\frac{943}{49^2}(-2\hat{i}+3\hat{j}+6\hat{k})$

37. $\vec{a}_1\cdot\vec{b} =$

(a) -41 (b) $-\frac{41}{7}$ (c) 41 (d) 287

38. Which of the following is true?

(a) $\vec{a}$ and $\vec{a}_2$ are collinear

(b) $\vec{a}_1$ and $\vec{c}$ are collinear

(c) $\vec{a},\vec{a}_1,\vec{b}$ are coplanar

(d) $\vec{a},\vec{a}_1,\vec{a}_2$ are coplanar

Paragraph for Question No. 39 to 42

Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are forming a right handed system, if $\vec{a}\times\vec{b}=\vec{c},\vec{b}\times\vec{c}=\vec{a},\vec{c}\times\vec{a}=\vec{b}$. If vectors $\vec{a},\vec{b}$ and $\vec{c}$ are forming a right handed system, then answer the following question.

39. If $\vec{x}=\vec{a}+\vec{b}-\vec{c}$, $\vec{y}=-\vec{a}+\vec{b}-2\vec{c}$, $\vec{z}=-\vec{a}+2\vec{b}-\vec{c}$, then a unit vector normal to the vector $\vec{x}+\vec{y}$ and $\vec{y}+\vec{z}$ is

(a) $\vec{a}$ (b) $\vec{b}$

(c) $\vec{c}$ (d) none of these

40. Vector $2\vec{a}-3\vec{b}+4\vec{c}$, $\vec{a}+2\vec{b}-\vec{c}$ and $x\vec{a}-\vec{b}+2\vec{c}$ are coplanar, then $x =$

(a) 8/5 (b) 5/8

(c) 0 (d) none of these

41. Let $\vec{x}=\vec{a}+\vec{b}$, $\vec{y}=2\vec{a}-\vec{b}$, then the point of intersection of straight lines $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}, \vec{r}\times\vec{y}=\vec{x}\times\vec{y}$ is

(a) $\frac{8}{5}$ (b) $\frac{5}{8}$

(c) $3\vec{a}$ (d) none of these

42. $\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})$ is equal to

(a) 1 (b) 3

(c) 0 (d) none of these

Paragraph for Question No. 43 to 46

The vertices of a triangle ABC are $A \equiv (2, 0, 2)$, $B \equiv (-1, 1, 1)$ and $C \equiv (1, -2, 4)$. The points D and E divide the side AB and CA in the ratio 1 : 2 respectively. Another point F is taken in space such that perpendicular drawn from F on ΔABC, meets the triangle at the point of intersection of the line segment CD and BE, say P. If the distance of F from the plane of the ΔABC is $\sqrt{2}$ units, then

43. The position vector of P, is

(a) $\hat{i}+\hat{j}+3\hat{k}$ (b) $\hat{i}-\hat{j}+3\hat{k}$

(c) $2\hat{i}-\hat{j}-3\hat{k}$ (d) none of these

44. The vector is

(a) $7\hat{j}+7\hat{k}$ (b) $\frac{7}{\sqrt{2}}(\hat{j}+\hat{k})$

(c) $(\hat{j}+\hat{k})$ (d) none of these

45. The volume of the tetrahedron $ABCF$, is

(a) 7 cubic units (b) 3/5 cubic units

(c) 7/3 cubic units (d) none of these

46. The equation of the line AF, is

(a) $\vec{r}=(2\hat{i}+2\hat{k})+\lambda(\hat{i}+2\hat{k})$

(b) $\vec{r}=(2\hat{i}+2\hat{k})+\lambda(\hat{i}-2\hat{k})$

(c) $\vec{r}=(\hat{i}+\hat{k})+\lambda(\hat{i}+2\hat{k})$

(d) $\vec{r}=(2\hat{i}+2\hat{k})+\lambda(-\hat{i}+2\hat{k})$

Paragraph for Question No. 47 to 49

Let a plane P_1 passes through the point (1, –2, 3) and is parallel to the plane P_2 given by $2x - 2y + z = 0$

47. The distance of the point (–1, 2, 0) from the plane P_1 is

(a) 2 units (b) 3 units (c) 5 units (d) 7 units

48. The coordinates of the foot of perpendicular drawn from point (1, –2, 3) to the plane P_2 are

(a) (0, 0, 0) (b) (–1, 0, 2)

(c) (1, 0, –2) (d) (2, 0, –4)

49. The distance between parallel planes P_1 and P_2 is

(a) 2 units (b) 3 units (c) 5 units (d) 7 units

SECTION-IV

Matrix-Match Type

50. Match the following.

	Column I		Column II
(A)	The area of the triangle whose vertices are the points, with rectangular cartesian coordinates (1, 2, 3), (–2, 1, –4), (3, 4, –2) is	(P)	0
(B)	The value of $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times d)+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})$ is	(Q)	1
(C)	A square $PQRS$ of side length p is folded along the diagonal PR so that planes PRQ and PRS are perpendicular to each other, the shortest distance between PQ and RS is, $\frac{p}{k\sqrt{2}}$ then $k =$	(R)	$\frac{\sqrt{1218}}{8}$
(D)	$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}, \vec{b}=-\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{d}=3\hat{i}+2\hat{j}+\hat{k}$ then $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=$	(S)	21

51. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at angle α to each other, then match the following.

	Column I		Column II
(A)	$\lvert\vec{a}+\vec{b}\rvert<1$ if	(P)	$\frac{2\pi}{3}<\alpha\le\pi$
(B)	$\lvert\vec{a}-\vec{b}\rvert=\lvert\vec{a}+\vec{b}\rvert$ if	(Q)	$\frac{\pi}{2}<\alpha\le\pi$
(C)	$\lvert\vec{a}+\vec{b}\rvert<\sqrt{2}$ if	(R)	$\alpha=\frac{\pi}{2}$
(D)	$\lvert\vec{a}-\vec{b}\rvert<\sqrt{2}$ if	(S)	$0\le\alpha<\frac{\pi}{2}$

52. Match the following

	Column I		Column II
(A)	If $\vec{a}=x\hat{i}+(x-1)\hat{j}+\hat{k}$ and $\vec{b}=(x+1)\hat{i}+\hat{j}+a\hat{k}$ always make an acute angle with each other for all $x\in R$, then number of non positive integral values of 'a' is	(P)	1
(B)	Let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{x}, \vec{a}\cdot\vec{x}=1, \vec{b}\cdot\vec{x}=\frac{3}{2}$, $\lvert\vec{x}\rvert=2$ and 'θ' is angle between $\vec{c}$ and $\vec{x}$ then $[2\cos\theta+2]$ is ($[\cdot]$denotes G. I. F)	(Q)	0
(C)	If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$, $\vec{c}=\hat{i}+p\hat{j}+q\hat{k}$ are linearly dependent and $\lvert\vec{c}\rvert=\sqrt{3}$, then $p^2-q^2=$	(R)	2
(D)	If $\vec{a},\vec{b},\vec{c}$ are non-coplanar and $\vec{a}+\vec{b}+\vec{c}=\alpha\vec{d}$, $\vec{b}+\vec{c}+\vec{d}=\beta\vec{a}$, then $\lvert\vec{a}+\vec{b}+\vec{c}+\vec{d}\rvert=$	(S)	3

53. Match the following.

	Column I		Column II
(A)	If in a cube, θ is the angle between any two body-diagonals then the value of $\cos\theta$ is	(P)	1
(B)	If in a cube, θ is the angle between a body-diagonal and a face-diagonal which is skew to it, then the value of $\sin\theta$ is	(Q)	$\frac{1}{\sqrt{2}}$
(C)	If in a cube, θ is the angle between diagonals of two faces through a vertex, then the value of $\cot\theta$ is	(R)	$\frac{1}{\sqrt{3}}$
(D)	If in a cube, θ is the angle between a body-diagonal and a face-diagonal interesting it then the value of $\tan\theta$ is	(S)	1/3

SECTION-V

Integer Answer Type

54. If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors and $[(\vec{a}+\vec{b})\times(\vec{b}-\vec{c})\ \ (\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\ \ (\vec{c}-\vec{a})\times(\vec{a}+\vec{b})]$ $=K[\vec{a}\ \vec{b}\ \vec{c}]^2$ then value of K is

55. $OABC$ is regular tetrahedron of unit edge length with volume V, then $12\sqrt{2}V=$

56. Find the distance of the point $\hat{i}+2\hat{j}+3\hat{k}$ from the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=5$ measured parallel to the vector $2\hat{i}+3\hat{j}-6\hat{k}$.

57. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$ then $\lvert\vec{a}+\vec{b}+\vec{c}\rvert$ is

58. Shortest distance between the z-axis and the line $x+y+2z-3=0=2x+3y+4z-4$ is

59. The equation of the plane passing through the intersection of the planes $2x-5y+z=3$ and $x+y+4z=5$ and parallel to the plane $x+3y+6z=1$ is $x+3y+6z=k$, where k is

60. A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-8/3}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If the distance $PQ=l$, then the value of $[l]$ is (where [.] represents the greatest integer function)

SOLUTIONS

1. (c) : Equation of plane passing through $\vec{a},\vec{b}$ and containing the line is $[\overline{AP}\ \overline{AB}\ \vec{c}]=0$

$\Rightarrow\ (\vec{r}-\vec{a})\cdot((\vec{b}-\vec{a})\times\vec{c})=0$

$\Rightarrow \vec{r}\cdot(\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=[\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore$ Length of perpendicular from the origin

$$=\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{\lvert\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\rvert}$$

2. (a) : Equation of any plane parallel to the given plane is $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+\lambda=0$.

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, we get $2x-3y+5z+\lambda=0$

This plane passes through the point $(3, 4, -1)$ if $2\times3-3\times4+5(-1)+\lambda=0$ or $\lambda=11$

Hence the equation of the required plane is

$r\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$

Rvp

3. **(d) :** Since the three vectors are coplanar. So,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ x & -2 & -1-x \end{vmatrix} = 0$$

$\Rightarrow \quad -2(-1 - x) + 2 = 0 \quad \Rightarrow \quad x = -2$

4. **(a)**

5. **(c) :** Equation of any plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ is $[(2x - 3y + 4z - 1) + \lambda(x - y + 4)] = 0$

$\Rightarrow (2 + \lambda)\,x - (3 + \lambda)\,y + 4z + 4\lambda - 1 = 0 \qquad \ldots(1)$

The plane (1) is perpendicular to the plane

$r\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$

So, $2(2 + \lambda) + (3 + \lambda) + 4 = 0$

$\Rightarrow \quad 11 + 3\lambda = 0 \Rightarrow \lambda = -11/3$

$\therefore$ The required equation of the plane is

$3(2x - 3y + 4z - 1) - 11(x - y + 4) = 0$

$\Rightarrow \quad 5x - 2y - 12z + 47 = 0$

6. **(b) :** We have $\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})=\vec{0}$

Taking dot product with $\vec{c}$, we have

$\alpha[\vec{a}\;\vec{b}\;\vec{c}]+\beta[\vec{b}\;\vec{c}\;\vec{c}]+\gamma[\vec{c}\;\vec{a}\;\vec{c}]=0$

$\Rightarrow \quad \alpha[\vec{a}\;\vec{b}\;\vec{c}]+0+0=0$

$\Rightarrow \quad \alpha[\vec{a}\;\vec{b}\;\vec{c}]=0$

Similarly, taking dot product with b and c, we have

$\beta[\vec{a}\;\vec{b}\;\vec{c}]=\gamma[\vec{a}\;\vec{b}\;\vec{c}]=0$

Now, even if one of $\alpha, \beta, \gamma \neq 0$, then we have $[\vec{a}\;\vec{b}\;\vec{c}]=0$

$\Rightarrow \quad \vec{a},\ \vec{b},\ \vec{c}$ are coplanar

7. **(c) :** $((\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}))\cdot(\vec{a}\times\vec{d})=0,$

$\Rightarrow \quad ([\vec{a}\;\vec{c}\;\vec{d}]\vec{b}-[\vec{b}\;\vec{c}\;\vec{d}]\vec{a})\cdot(\vec{a}\times\vec{d})=0$

$\Rightarrow [\vec{a}\;\vec{c}\;\vec{d}][\vec{b}\;\vec{a}\;\vec{d}]=0$

$\Rightarrow$ Either $\vec{c}$ or $\vec{b}$ must lie in the plane of $\vec{a}$ and $\vec{d}$.

8. **(c) :** $\vec{r}=(\vec{a}\times\vec{b})\sin x+(\vec{b}\times\vec{c})\cos y+2(\vec{c}\times\vec{a})$

$\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow \quad [\vec{a}\;\vec{b}\;\vec{c}](\sin x+\cos y+2)=0$

Now, $[\vec{a}\;\vec{b}\;\vec{c}]\neq 0 \Rightarrow \sin x+\cos y=-2$

This is possible only when $\sin x = -1$ and $\cos y = -1$

for $x^2 + y^2$ to be minimum $x = -\dfrac{\pi}{2}$ and $y=\pi$

$\Rightarrow$ minimum value of $(x^2 + y^2)$ is $\dfrac{\pi^2}{4}+\pi^2=\dfrac{5\pi^2}{4}$

9. **(b)**

10. **(c) :** Consider $OABC$, $\overrightarrow{OA}=\vec{\alpha}, \overrightarrow{OB}=\vec{\beta}, \overrightarrow{OC}=\vec{\gamma}$

And OA, BC as a pair of opposite edges.

$|\overrightarrow{OA}|=a, |\overrightarrow{BC}|=b$

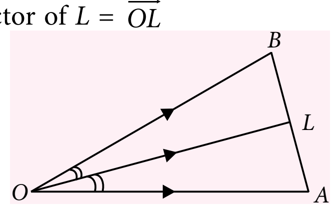
Equation of OA is $\vec{r}=\vec{O}+t\vec{\alpha}$

Equation of BC is $\vec{r}=\vec{\beta}+s(\vec{\beta}-\vec{\gamma})$

$$d=\frac{-[\vec{\alpha}\;\vec{\beta}\;\vec{\gamma}]}{|\vec{\alpha}||\vec{\beta}-\vec{\gamma}|\sin\theta}\Rightarrow V=\frac{abd\sin\theta}{6}$$

11. **(b) :** $|\overrightarrow{OA}|=3, |\overrightarrow{OB}|=6$

Position vector of $L = \overrightarrow{OL}$



$$=\frac{|\overrightarrow{OA}|(2\hat{i}+4\hat{j}+4\hat{k})+|\overrightarrow{OB}|(2\hat{i}+2\hat{j}+\hat{k})}{|\overrightarrow{OA}|+|\overrightarrow{OB}|}$$

$$=\frac{3(2\hat{i}+4\hat{j}+4\hat{k})+6(2\hat{i}+2\hat{j}+\hat{k})}{3+6}=\frac{18\hat{i}+24\hat{j}+18\hat{k}}{9}$$

$$=\frac{1}{3}(6\hat{i}+8\hat{j}+6\hat{k})$$

So, $|\overrightarrow{OL}|=\dfrac{1}{3}\sqrt{36+64+36}=\dfrac{\sqrt{136}}{3}$ units

12. **(b) :** Equation of the plane bisecting the angle between the given planes are

$$\frac{2x-y+2z+3}{\sqrt{(2)^2+(-1)^2+2^2}}=\pm\frac{3x-2y+6z+8}{\sqrt{3^2+(-2)^2+6^2}}$$

$\Rightarrow \quad 7(2x - y + 2z + 3) = \pm\, 3(3x - 2y + 6z + 8)$

$\Rightarrow \quad 5x - y - 4z - 3 = 0$ taking the +ve sign, and

$23x - 13y + 32z + 45 = 0$ taking the – ve sign.

13. **(c) :** The perpendicular distance of the origin (0, 0, 0) from the plane $x + y + z = p$ is

$$\left|\frac{-p}{\sqrt{1+1+1}}\right|=\frac{|p|}{\sqrt{3}}$$

If the coordinates of P are (l, m, n), then we must have

$$\left|\frac{l+m+n-p}{\sqrt{3}}\right|=\frac{|p|}{\sqrt{3}} \quad\Rightarrow\quad |l + m + n - p| = |p|$$

which is satisfied by (c)

14. **(c) :** Since the line makes an angle θ with the plane. So, it makes an angle $\pi/2 - \theta$ with normal to the plane

$\therefore \cos\left(\frac{\pi}{2}-\theta\right)=\frac{2(1)+(-1)(2)+(\sqrt{\lambda})(2)}{\sqrt{1+4+4}\times\sqrt{4+1+\lambda}}$

$\Rightarrow \frac{1}{3}=\frac{2\sqrt{\lambda}}{3\sqrt{\lambda+5}} \Rightarrow \lambda+5=4\lambda \Rightarrow \lambda = 5/3$

15. (b)

16. (d) : $A(7 + 3\alpha, 5 + 2\alpha, 3 + \alpha)$,
$B(1 + 2\beta, -1 + 4\beta, -1 + 3\beta)$

Dr's of AB are 2 : 2 : 1

$\therefore \frac{6+3\alpha-2\beta}{2}=\frac{3+\alpha-2\beta}{1}=\frac{4+\alpha-3\beta}{1}$

$\Rightarrow \alpha = -2, \beta = 1$

$\therefore A(1,1,1), B(3,3,2), \quad AB = 3$

17. (d) : Given planes are

$x - y - z = 4$ ... (1)

and $x + y + 2z = 4$ ... (2)

Since required plane passes through the line of intersection of (1) & (2)

$\therefore$ Its equation is

$(x - y - z - 4) + \alpha(x + y + 2z - 4) = 0$

$\Rightarrow (1 + \alpha)x + (\alpha - 1)y + (2\alpha - 1)z - (4\alpha + 4) = 0$ .... (3)

Since (1) & (3) are perpendicular

$\Rightarrow 1(1 + \alpha) - 1(\alpha - 1) -1(2\alpha - 1) = 0$

$\Rightarrow 1 + \alpha - \alpha + 1 -2\alpha + 1 = 0 \Rightarrow \alpha = 3/2$

$\Rightarrow$ Required equation is

$(x - y - z - 4)+\frac{3}{2}(x + y + 2z - 4) = 0$

$\Rightarrow 5x + y + 4z - 20 = 0$

18. (b) : Coordinates of any point Q on the given line are $(2r + 1, -3r - 1, 8r - 10)$ for some $r \in R$

So the direction ratios of PQ are $2r, -3r - 1, 8r - 10$

Now PQ is perpendicular to the given line

if $2(2r) - 3(-3r - 1) + 8(8r - 10) = 0$

$\Rightarrow 77r - 77 = 0 \Rightarrow r = 1$

$\therefore$ The coordinates of Q *i.e.*, the foot of the perpendicular from P on the line are $(3, -4, -2)$.

Let $R(a, b, c)$ be the reflection of P in the given line when Q is the mid-point of PR

$\Rightarrow \frac{a+1}{2}=3, \frac{b}{2}=-4, \frac{c}{2}=-2$

$\Rightarrow a = 5, b = -8, c = -4$

and the coordinates of the required point are $(5, -8, -4)$.

19. (c) : Vector along the required plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\hat{i}-\hat{j}-10\hat{k}$$

So, normal vector $(\vec{n})$ to the plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -1 & -10 \\ 2 & 3 & 4 \end{vmatrix} = 26\hat{i}-52\hat{j}+26\hat{k}.$$

So, equation of the plane is $\vec{r}\cdot\vec{n}=0 \Rightarrow x-2y+z=0.$

20. (d) **21. (d)**

22. (c) : We have,

$$\Delta^2 = \Delta\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1 \Rightarrow |\Delta| = 1$$

23. (a, c) : Given $\vec{v}=\vec{a}\times\vec{b} \Rightarrow |\vec{v}|=|\vec{a}||\vec{b}|\sin\theta=\sin\theta$

$\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}=\vec{a}-\vec{b}\cos\theta$

$\Rightarrow |\vec{u}|^2=(\vec{a}-\vec{b}\cos\theta)^2=|\vec{a}|^2+|\vec{b}|^2\cos^2\theta-2\vec{a}\cdot\vec{b}\cos\theta$

$=1+\cos^2\theta-2\cos^2\theta=1-\cos^2\theta=\sin^2\theta \Rightarrow |\vec{u}|=|\vec{v}|$

Again $\vec{u}\cdot\vec{b}=\vec{a}\cdot\vec{b}-(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b})=0 \Rightarrow |\vec{u}\cdot\vec{b}|=0$

So, $|\vec{v}|=|\vec{u}|+|\vec{u}\cdot\vec{b}|$

24. (a, d) : (a) is proved if we take dot product of both sides with $\vec{a}$.

(b) If we take dot product with $\vec{b}$, we get

$\lambda_2=\vec{b}\cdot\vec{c}$

$\Rightarrow$ Choice (b) is not true.

(c) If we take dot product of both sides with $\vec{a}\times\vec{b}$, we get $[\vec{c}\,\vec{b}\,\vec{a}]=\lambda_3[\vec{a}\times\vec{b}]^2$

$\Rightarrow \lambda_3=[\vec{a}\,\vec{b}\,\vec{c}]$ or $\vec{c}\cdot(\vec{a}\times\vec{b})$

$\Rightarrow$ Choice (c) is wrong.

(d) is correct since $\lambda_1+\lambda_2+\lambda_3=\vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c}+[\vec{a}\,\vec{b}\,\vec{c}]$.

25. (a, b, c) : $\vec{r}\times(\hat{i}+2\hat{j}+\hat{k})=\hat{i}-\hat{k}$

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore\ (x\hat{i}+y\hat{j}+z\hat{k})\times(\hat{i}+2\hat{j}+\hat{k})=\hat{i}-\hat{k}\Rightarrow\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\1&2&1\end{vmatrix}=\hat{i}-\hat{k}$$

Put values from options and check.

26. (a, b, d) :

$$(\lambda-1)(\vec{a}_1-\vec{a}_2)+\mu(\vec{a}_2+\vec{a}_3)+\gamma(\vec{a}_3+\vec{a}_4-2\vec{a}_2)+\vec{a}_3+\delta\vec{a}_4=\vec{0}$$

$$i.e., (\lambda-1)\vec{a}_1+(1-\lambda+\mu-2\gamma)\vec{a}_2+(\mu+\gamma+1)\vec{a}_3+(\gamma+\delta)\vec{a}_4=0$$

Since $\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4$ are linearly independent

$\therefore\ \lambda-1=0,\ 1-\lambda+\mu-2\gamma=0,\ \mu+\gamma+1=0$ and $\gamma+\delta=0$

i.e. $\lambda=1,\ \mu=2\gamma,\ \mu+\gamma+1=0,\ \gamma+\delta=0$

i.e. $\lambda=1,\ \mu=-\dfrac{2}{3},\gamma=-\dfrac{1}{3},\delta=\dfrac{1}{3}$

27. (a, c, d)

28. (a, d) : $\because\ \vec{r}\cdot\vec{a}=10$

$\therefore\ x+y+z=10;\ x\ge 1,\ y\ge 1,\ z\ge 1$

The required number of positions

$={}^{10-1}C_{3-1}={}^{9}C_2=36$

29. (a, b) : Equation of a plane passing through the line $3x-y+2z-1=0=x+2y-z-2$ is

$3x-y+2z-1+\lambda(x+2y-z-2)=0$

Since it is perpendicular to the given plane

$\therefore\ \lambda=-\dfrac{3}{2}$

$\therefore$ Equation of the line of projection is

$3x-8y+7z+4=0=3x+2y+z$

Its direction ratios are $<11,\ -9,\ -15>$ and the point $(-1,1,1)$ lies on the line

$\therefore\ \dfrac{x+1}{11}=\dfrac{y-1}{-9}=\dfrac{z-1}{-15}$ is also the equation of the line of projection.

30. (a, c)

31. (a, b): Mid point of $BC=\left(\dfrac{\lambda-1}{2},4,\dfrac{\mu+2}{2}\right)$

d.r's of median through A are

$$\left(\frac{\lambda-1}{2}-2,4-3,\frac{\mu+2}{2}-5\right)=\left(\frac{\lambda-5}{2},1,\frac{\mu-8}{2}\right)$$

The median is equally inclined to axes, so the direction ratios must be equal, so

$$\frac{\lambda-5}{2}=1=\frac{\mu-8}{2}\Rightarrow\lambda=7,\mu=10$$

32. (a, b) : $3x-6y+2z+5=0$ ...(i)

$-4x+12y-3z+3=0$ ...(ii)

$$\frac{3x-6y+2z+5}{\sqrt{9+36+4}}=\frac{-4x+12y-3z+3}{\sqrt{16+144+9}}$$

Bisects the angle between the planes that contains the origin

$\Rightarrow\ 13(3x-6y+2z+5)=7(-4x+12y-3z+3)$

$\Rightarrow\ 39x-78y+26z+65=-28x+84y-21z+21$

$\Rightarrow\ 67x-162y+47z+44=0$ ...(iii)

Further $3(-4)+(-6)(12)+2(-3)<0$

$\therefore$ Origin lies in acute angle

33. (a, b) : Equation of required plane is

$lx+my+\lambda z=0$ ...(1)

Angle between (1) and $lx+my=0$ is α.

$$\Rightarrow\ \cos\alpha=\frac{l^2+m^2}{\sqrt{l^2+m^2}\sqrt{l^2+m^2+\lambda^2}}$$

$$\Rightarrow\cos^2\alpha=\frac{l^2+m^2}{l^2+m^2+\lambda^2}\ \Rightarrow\ \lambda=\pm\sqrt{l^2+m^2}\tan\alpha$$

Hence equation of plane is

$$lx+my\pm z\sqrt{l^2+m^2}\tan\alpha=0$$

34. (c) : The required plane is

$x+2y+z-3+k(2x-y+z-5)=0$

Since it passes through A,

$\therefore\ k=1/3$

$\therefore$ The equation of plane is $5x+5y+4z-14=0$,

i.e. $\vec{r}\cdot(5\hat{i}+5\hat{j}+4\hat{k})=14$

35. (b) : The line of intersection of planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is parallel to $\vec{n}_1\times\vec{n}_2$.

36. (b) :

$$\vec{a}_1=\left[(2\hat{i}+3\hat{j}-6\hat{k})\cdot\frac{(2\hat{i}-3\hat{j}+6\hat{k})}{7}\right]\left(\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7}\right)$$

$$=\frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})$$

$$\vec{a}_2=\frac{-41}{49}\left((2\hat{i}-3\hat{j}+6\hat{k})\cdot\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right)\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$=\frac{-41}{(49)^2}(-4-9+36)(-2\hat{i}+3\hat{j}+6\hat{k})$$

$$=\frac{943}{(49)^2}(2\hat{i}-3\hat{j}-6\hat{k})$$

37. (a) : $\vec{a}_1\cdot\vec{b}=\frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k})=-41$

38. (c) : $\vec{a},\vec{a}_1,\vec{b}$ are coplanar, because $\vec{a}_1$, $\vec{b}$ are collinear.

39. (d) : $\vec{x}+\vec{y}=2\vec{b}-3\vec{c}$ and $\vec{y}+\vec{z}=-2\vec{a}+3\vec{b}-3\vec{c}$

$$\therefore\quad(\vec{x}+\vec{y})\times(\vec{y}+\vec{z})=\begin{vmatrix}\vec{a}&\vec{b}&\vec{c}\\0&2&-3\\-2&3&-3\end{vmatrix}=3\vec{a}+6\vec{b}+4\vec{c}$$

Required unit vector $=\dfrac{3\vec{a}+6\vec{b}+4\vec{c}}{\sqrt{61}}$

40. (a) : $\begin{vmatrix}2&-3&4\\1&2&-1\\x&-1&2\end{vmatrix}=0$

$\Rightarrow$ 2(4 – 1) + 3(2 + x) + 4(–1 – 2x) = 0 $\Rightarrow$ $x=\frac{8}{5}$

41. (c) : $\vec{r}\times\vec{x}=\vec{y}\times\vec{x}\Rightarrow(\vec{r}-\vec{y})\times\vec{x}=\vec{0}$
$\Rightarrow\ \vec{r}=\vec{y}+\lambda\vec{x}$

Also, $\vec{r}\times\vec{y}=\vec{x}\times\vec{y}\Rightarrow(\vec{r}-\vec{x})\times\vec{y}=\vec{0}\Rightarrow\vec{r}=\vec{x}+\mu\vec{y}$
Now, $\vec{y}+\lambda\vec{x}=\vec{x}+\mu\vec{y}$
$\Rightarrow\ (2\vec{a}-\vec{b})+\lambda(\vec{a}+\vec{b})=(\vec{a}+\vec{b})+\mu(2\vec{a}-\vec{b})$
$\Rightarrow$ 2 + λ = 1 + 2μ, –1 + λ = 1 – μ $\Rightarrow$ μ = 1, λ = 1
$\therefore$ The point of intersection is $3\vec{a}$.

42. (b): $\vec{a}\times\vec{b}=\vec{c}\Rightarrow\vec{c}\cdot(\vec{a}\times\vec{b})=\vec{c}\cdot\vec{c}=1\Rightarrow[\vec{a}\ \vec{b}\ \vec{c}]=1$
$\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})=3$

43–46

43. (b) 44. (c) 45. (c) 46. (d)

The vector equation of CD and BE are

$$\vec{r}=(\hat{i}-2\hat{j}+4\hat{k})+\frac{\lambda}{3}(7\hat{j}-7\hat{k})\qquad\text{....(i)}$$

and $\vec{r}=(-\hat{i}+\hat{j}+\hat{k})+\frac{\mu}{3}(7\hat{i}-7\hat{j}+7\hat{k})$ ....(ii)

respectively.
CD and BE intersect at point P. At their point of intersection, we must have

$$(\hat{i}-2\hat{j}+4\hat{k})+\frac{\lambda}{3}(7\hat{j}-7\hat{k})=(-\hat{i}+\hat{j}+\hat{k})+\frac{\mu}{3}(7\hat{i}-7\hat{j}+7\hat{k})$$

$\Rightarrow\ 1=-1+\frac{7\mu}{3},\ -2+\frac{7\lambda}{3}=1-\frac{7\mu}{3}$

and $4-\frac{7\lambda}{3}=1+\frac{7\mu}{3}\ \Rightarrow\ \mu=\frac{6}{7}$ and $\lambda=\frac{3}{7}$

Substituting the value of λ in (i) or that of μ in (ii), we obtain the position vector $\vec{r}$ of point P as,
$\vec{r}=\hat{i}-\hat{j}+3\hat{k}$

Now, $\Delta=$ area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$=\frac{1}{2}|(-3\hat{i}+\hat{j}-\hat{k})\times(-\hat{i}-2\hat{j}+2\hat{k})|=\frac{7\sqrt{2}}{2}\text{ sq. unit}$$

$\therefore$ Volume of the tetrahedron $ABCF$

$=\frac{1}{3}\times$ (area of the base) $\times$ height

$=\frac{1}{3}\cdot\frac{7\sqrt{2}}{2}\cdot\sqrt{2}=\frac{7}{3}$ cubic units

We have, $\overrightarrow{AB}\times\overrightarrow{AC}=7\hat{j}+7\hat{k}$

Now, $\overrightarrow{PF}=\sqrt{2}\,\frac{(7\hat{j}+7\hat{k})}{\sqrt{49+49}}=\hat{j}+\hat{k}$

$\Rightarrow\ \overrightarrow{PF}=\hat{j}+\hat{k}$

$\Rightarrow$ Position vector of $\vec{F}=(\hat{j}+\hat{k})+(\hat{i}-\hat{j}+3\hat{k})$
$=\hat{i}+4\hat{k}$

$\therefore$ Vector equation of AF is,
$\vec{r}=(2\hat{i}+2\hat{k})+\lambda(\hat{i}+4\hat{k}-2\hat{i}-2\hat{k})$
$\vec{r}=(2\hat{i}+2\hat{k})+\lambda(-\hat{i}+2\hat{k})$

47. (c) 48. (b) 49. (b)

50. A - R; B - P; C - Q; D - S

(A) $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k},\ \overrightarrow{OB}=-2\hat{i}+\hat{j}-4\hat{k},$
$\overrightarrow{OC}=3\hat{i}+4\hat{j}-2\hat{k}$

Area $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{\sqrt{1218}}{2}$

(B) $((\vec{a}\times\vec{b})\times\vec{c})\cdot\vec{d}+((\vec{b}\times\vec{c})\times\vec{a})\cdot\vec{d}+((\vec{c}\times\vec{a})\times\vec{b})\cdot\vec{d}=0$
(D) $(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})=21$

51. A – P; B – R; C - Q, P; D – S

(A) If $|\vec{a}+\vec{b}|<1$ then $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})<1$

So $|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}<1\Rightarrow\vec{a}\cdot\vec{b}<-\frac{1}{2}$

$\Rightarrow\ \cos\alpha<-\frac{1}{2}\ \Rightarrow\ \frac{2\pi}{3}<\alpha\leq\pi$

(B) If $|\vec{a}-\vec{b}|=|\vec{a}+\vec{b}|$, then $\vec{a}\cdot\vec{b}=0 \Rightarrow \alpha=\frac{\pi}{2}$

(C) If $|\vec{a}+\vec{b}|<\sqrt{2}$, then $\cos\alpha < 0$ which is true if $\frac{\pi}{2}<\alpha\le\pi$

(D) If $|\vec{a}-\vec{b}|<\sqrt{2}$, then $\cos\alpha > 0$ which is true if $0 \le \alpha < \pi$.

52. A - Q; B - S; C - Q; D - Q

(A) $\vec{a}\cdot\vec{b}>0 \Rightarrow x^2+2x+a-1>0$

$\Rightarrow \Delta<0 \quad \Rightarrow \quad a>2$

(B) $(\vec{a}+\vec{b}+\vec{c})\cdot\vec{x}=\vec{x}\cdot\vec{x} \Rightarrow 1+\frac{3}{2}+\vec{c}\cdot x=4 \Rightarrow \vec{c}\cdot\vec{x}=\frac{3}{2}$

$\Rightarrow |\vec{c}||\vec{x}|\cos\theta=\frac{3}{2} \Rightarrow \cos\theta=\frac{3}{4} \Rightarrow [2\cos\theta+2]=2$

(C) $\begin{vmatrix}1 & 1 & 1\\ 4 & 3 & 4\\ 1 & p & q\end{vmatrix}=0 \Rightarrow q=1.$ Also, $|\vec{c}|=\sqrt{3} \Rightarrow p^2=1$

Hence $p^2 - q^2 = 0$.

53. A – S; B – P; C – R; D – Q

54. (4) : $[(\vec{a}+\vec{b})\times(\vec{b}-\vec{c})\ (\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\ (\vec{c}-\vec{a})\times(\vec{a}+\vec{b})]$

$=[\vec{a}\times\vec{b}-\vec{b}\times\vec{c}+\vec{c}\times\vec{a} \quad -\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$
$\quad -\vec{a}\times\vec{b}-\vec{b}\times\vec{c}+\vec{c}\times\vec{a}]$

$$=[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]\begin{vmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ -1 & -1 & 1\end{vmatrix}=4[\vec{a}\,\vec{b}\,\vec{c}]^2$$

55. (2) : $$\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}=\begin{vmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1\end{vmatrix}=\frac{1}{2}$$

$\Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]=\frac{1}{\sqrt{2}}$

Volume $=\frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\frac{1}{6\sqrt{2}}$

$\therefore \quad 12\sqrt{2}V=2$

56. (7) : The distance of the point 'a' from the plane $\vec{r}\cdot\vec{n}=q$ measured in the direction of the unit vector

$\vec{b}=\dfrac{q-\vec{a}\cdot\vec{n}}{\hat{b}\cdot\vec{n}}$

Here $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{n}=\hat{i}+\hat{j}+\hat{k}$ and $q = 5$

Also $\hat{b}=\dfrac{2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(2)^2+(3)^2+(-6)^2}}=\dfrac{2\hat{i}+3\hat{j}-6\hat{k}}{7}$

$\therefore$ The required distance

$$=\left|\frac{5-(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}\right|=\left|\frac{5-(1+2+3)}{\frac{1}{7}(2+3-6)}\right|=7$$

57. (2) : $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\cdot\vec{c}=0$

$\vec{b}\cdot\vec{c}=|\vec{b}||\vec{c}|\cos\frac{\pi}{3}=\frac{1}{2}.$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2=3+2\cdot0+2\cdot0+1=4$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|=2$

58. (2) : Equation of plane

$x + y + 2z - 3 + \lambda(2x + 3y + 4z - 4) = 0$ ....(1)

If plane (1) is parallel to z-axis $\Rightarrow \lambda=-\frac{1}{2}$

Therefore plane, parallel to z-axis is $y + 2 = 0$ ....(2)

Now, shortest distance between any point on z-axis (0, 0, 1) (say) from plane (2) is 2

59. (7) : Equation of plane passing through the intersection of the planes $2x - 5y + z = 3$ and $x + y + 4z = 5$ is

$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$

$\Rightarrow (2 + \lambda)x + (-5 + \lambda)y + (1 + 4\lambda)z - 3 - 5\lambda = 0$ ....(i)

which is parallel to the plane $x + 3y + 6z = 1$.

Then, $\frac{2+\lambda}{1}=\frac{-5+\lambda}{3}=\frac{1+4\lambda}{6}$

$\therefore \lambda=\frac{-11}{2}$

From (i), we have

$-\frac{7}{2}x-\frac{21}{2}y-21z+\frac{49}{2}=0$

$\therefore \quad x + 3y + 6z = 7$

Hence, $k = 7$

60. (2) : From the given conditions, we have,

$\frac{2\mu+8/3}{\lambda+2}=\frac{\mu+3}{2\lambda-1}=\frac{\mu+1}{\lambda-1} \quad \Rightarrow \lambda=3, \mu=\frac{1}{3}$

$\Rightarrow \quad P\equiv(5,-5,2),\ Q\equiv\left(\frac{10}{3},\frac{-10}{3},\frac{4}{3}\right)$

Now, $l = PQ = \sqrt{6} \Rightarrow [l] = 2$

WB JEE MOCK TEST PAPER

The entire syllabus of Mathematics of WB-JEE is being divided into six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:
Unit- I: Algebra, Unit-II: Trigonometry, Unit-III: Co-ordinate geometry of two dimensions & three dimensions, Unit-IV: Calculus, Unit-V: Vector, Unit-VI: Statistics & Probability.

1. An ellipse has eccentricity 1/2 and one focus at $S\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent (nearer to S) to the circle $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. The equation of the ellipse in standard form is

(a) $9\left(x-\frac{1}{3}\right)^2 + 12(y-1)^2 = 1$

(b) $12\left(x-\frac{1}{3}\right)^2 + 9(y-1)^2 = 1$

(c) $\frac{\left(x-\frac{1}{2}\right)^2}{12} + \frac{(y-1)^2}{9} = 1$

(d) $3\left(x+\frac{1}{2}\right)^2 + 4(y-1)^2 = 1$

2. AB is a chord of the parabola $y^2 = 4ax$ with vertex at A. BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the x-axis is

(a) a (b) $2a$
(c) $4a$ (d) $8a$

3. If both roots of $x^2 - 2ax + a^2 - 1 = 0$ lies in $(-2, 1)$ then $[a]$, where $[\cdot]$ denotes greatest integral function is

(a) -1 (b) 0
(c) 1 (d) 2

4. Let R be a relation in N defined by $R = \{(x, y) : 2x + y = 8\}$, then range of R is

(a) $\{1, 2, 3\}$ (b) $\{2, 4, 6\}$
(c) $\{1, 2, 3, 4, 6\}$ (d) none of these

5. Let $f(x) = \frac{\sin 4\pi[x]}{1+[x]^2}$, where $[x]$ is the greatest integer, then

(a) $f(x)$ is not differentiable
(b) $f'(x) > 0$
(c) $f'(x) = 0 \ \forall \ x$
(d) none of these

6. If $z \neq 0$, $\int_0^{100} \arg(-|z|)dx$ equals

(a) 0 (b) not defined
(c) 100 (d) 100π

7. Sum of the series

$$\frac{1}{1!(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \dots + \frac{1}{(n-1)!1!}$$ is

(a) $\frac{2^{n-1}}{(n-1)!}$ (b) $\frac{2^{n-1}}{n!}$

(c) $\frac{2^n}{(n-1)!}$ (d) $\frac{2^n}{n!}$

8. In a 'multiple choice question' test there are eight questions. Each question has four alternatives of which only one is correct. If a candidate answers all the questions by choosing one answer for each question, then the number of ways to get exactly 4 correct answer is

(a) 70 (b) 2835
(c) 5670 (d) none of these

9. There are two girls Rima and Tina having m and n distinct number of toys respectively. The number of

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ways in which they can exchange their toys so that after exchange they have the same number of toys with them but not the same toys is

(a) ${}^{m+n}C_m$ (b) ${}^{m+n}C_m - 1$
(c) ${}^{m+n}P_n$ (d) none of these

10. If the roots of $x^2 + 2ax + b = 0$ are real and they differ by almost $2m$, then b lies in the interval

(a) $(a^2 - m^2, a^2)$ (b) $(a^2, a^2 + m^2)$
(c) $[a^2 - m^2, a^2]$ (d) $(a^2 - m^2, a^2 + m^2)$

11. If $\frac{1+3p}{4}, \frac{1-p}{3}, \frac{1-3p}{2}$ are the probabilities of three mutually exclusive events, then the set of all values of p is

(a) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (b) $\left[-\frac{1}{3}, 1\right]$
(c) $\left[\frac{1}{13}, 1\right]$ (d) $\left[\frac{1}{13}, \frac{1}{3}\right]$

12. $f(x) = \begin{cases} \frac{3[x]-5|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$. Then $\int_{-3/2}^{2} f(x)dx =$

([·] denotes the greatest integer function)

(a) $-\frac{11}{2}$ (b) $-\frac{7}{2}$ (c) -6 (d) $-\frac{17}{2}$

13. In a triangle $\angle A > \angle B$ and $\angle A$ and $\angle B$ satisfy the equation $3\sin\theta - 4\sin^3\theta - k = 0$, $0 < k < 1$. Then $\angle C =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

14. Let $f : N \rightarrow Y$ be a fuction defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$ such that f is invertible, then its inverse is

(a) $g(y) = 4 + \frac{y+3}{4}$ (b) $g(y) = \frac{y+3}{4}$

(c) $g(y) = \frac{3y+4}{3}$ (d) $g(y) = \frac{y-3}{4}$

15. Area bounded by $|x - 1| \leq 2$ and $x^2 - y^2 = 1$ is

(a) $6\sqrt{3} + \frac{1}{2}\log|3 + 2\sqrt{2}|$

(b) $6\sqrt{2} + \frac{1}{2}\log|3 - 2\sqrt{2}|$

(c) $6\sqrt{2} - \log|3 + 2\sqrt{2}|$

(d) none of these

16. Numbers 1, 2, 3, ..., $2n$ ($n \in N$) are printed on $2n$ cards. The probability of drawing a number r is proportional to r. Then the probability of drawing an even number in one draw is

(a) $\frac{n+2}{n+3}$ (b) $\frac{n+1}{n+3}$

(c) $\frac{1}{2}$ (d) $\frac{n+1}{2n+1}$

17. It is given that event A and B are such that $P(A) = \frac{1}{4}$, $P(B|A) = \frac{2}{3}$, $P(A|B) = \frac{1}{2}$, then $P(B) =$

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

18. From the matrix equation $AB = AC$, we say $B = C$ provided

(a) A is singular
(b) A is skew symmetric
(c) A is non-singular
(d) none of these

19. If $f(x) = \begin{vmatrix} \sin x & 1 & 0 \\ 1 & 2\sin x & 1 \\ 0 & 1 & 2\sin x \end{vmatrix}$ then $\int_{-\pi/2}^{\pi/2} f(x)dx =$

(a) 0 (b) -1 (c) 1 (d) $\frac{3\pi}{2}$

20. The given expression

$$f(x) = \frac{1}{\tan x + \cot x + \sec x + \text{cosec} x}$$

is equivalent to

(a) $\frac{1}{2(\sin x + \cos x - 1)}$ (b) $\frac{\sin x + \cos x - 1}{2}$

(c) $\frac{1}{2(\sin x - \cos x + 1)}$ (d) $\frac{\sin x - \cos x + 1}{2}$

21. Let $f(x) = e^{\cos^{-1}\sin\left(x + \frac{\pi}{3}\right)}$, then $f\left(\frac{8\pi}{9}\right) =$

(a) $e^{\frac{7\pi}{12}}$ (b) $e^{\frac{13\pi}{18}}$ (c) $e^{\frac{5\pi}{18}}$ (d) $e^{\frac{\pi}{12}}$

22. In a triangle if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P. then a, b, c are in

(a) G.P. (b) H.P (c) A.P. (d) A.G.P.

23. If $\omega = \dfrac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$, then z lies on
(a) a circle (b) an ellipse
(c) a parabola (d) a straight line

24. If $g(x) = \int x^x \log_e(ex)dx$, then $g'(\pi)$ Equals
(a) $\pi\pi \log_e(e\pi)$ (b) $\pi \log_e \pi$
(c) $\pi^\pi \log_e \pi$ (d) none of these

25. If $xy = e - e^y$, then $\left(\dfrac{d^2y}{dx^2}\right)_{x=0}$ is
(a) $\dfrac{1}{e}$ (b) $\dfrac{1}{e^3}$
(c) $\dfrac{1}{e^2}$ (d) none of these

26. For any vector $\vec{a}$, the value of $(\vec{a}\times\hat{i})^2 + (\vec{a}\times\hat{j})^2 + (\vec{a}\times\hat{k})^2$ is equal to
(a) $|\vec{a}|^2$ (b) $3|\vec{a}|^2$
(c) $4|\vec{a}|^2$ (d) $2|\vec{a}|^2$

27. Let $\vec{a} = \hat{i} - \hat{k}$, $\vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}$ and $\vec{c} = y\hat{i} + x\hat{j} + (1 + x - y)\hat{k}$. Then $[\vec{a}\ \vec{b}\ \vec{c}]$ depends on
(a) x only (b) y only
(c) neither x nor y (d) both x and y

28. A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is
(a) $\log_3 e$ (b) $\log_e 3$
(c) $2\log_3 e$ (d) $\dfrac{1}{2}\log_e 3$

29. If $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is
(a) $x\log\dfrac{y}{x} = cy$ (b) $y\log\left(\dfrac{x}{y}\right) = cx$
(c) $\log\left(\dfrac{x}{y}\right) = cy$ (d) $\log\left(\dfrac{y}{x}\right) = cx$

30. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is
(a) $\dfrac{1}{18\pi}$ cm/min (b) $\dfrac{1}{36\pi}$ cm/min
(c) $\dfrac{5}{6\pi}$ cm/min (d) $\dfrac{1}{54\pi}$ cm/min

31. $\displaystyle\int\left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx =$
(a) $\dfrac{x}{x^2+1} + C$ (b) $\dfrac{\log x}{(\log x)^2 + 1} + C$
(c) $\dfrac{x}{(\log x)^2 + 1} + C$ (d) $\dfrac{xe^x}{1 + x^2} + C$

32. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$. Then there is a circle passing through P, Q and (1, 1) for
(a) all except one value of p
(b) all except two values of p
(c) exactly one value of p
(d) all values of p

33. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is –1 is
(a) $\dfrac{x}{2} + \dfrac{y}{3} = 1$ and $\dfrac{x}{2} + \dfrac{y}{1} = 1$
(b) $\dfrac{x}{2} - \dfrac{y}{3} = -1$ and $\dfrac{x}{-2} + \dfrac{y}{1} = 1$
(c) $\dfrac{x}{2} + \dfrac{y}{3} = -1$ and $\dfrac{x}{-2} + \dfrac{y}{1} = 1$
(d) $\dfrac{x}{2} - \dfrac{y}{3} = 1$ and $\dfrac{x}{-2} + \dfrac{y}{1} = 1$

34. $\dfrac{x+4}{x-3} < 2$ is satisfied when x satisfies
(a) $(-\infty, 3) \cup (10, \infty)$ (b) (3, 10)
(c) $(-\infty, 3) \cup [10, \infty)$ (d) none of these

35. The set of values of x which satisfy the inequations $5x + 2 < 3x + 8$ and $\dfrac{x+2}{x-1} < 4$, $x \neq 1$ is
(a) $(-\infty, 1)$ (b) (2, 3)
(c) $(-\infty, 3)$ (d) $(-\infty, 1) \cup (2, 3)$

36. If $\dfrac{4^n}{n+1} < \dfrac{(2n)!}{(n!)^2}$, then $P(n)$ is true for
(a) $n \geq 1$ (b) $n > 0$
(c) $n < 0$ (d) $n \geq 2$

37. If $n(U) = 700$, $n(A) = 200$, $n(B) = 240$, $n(A \cap B) = 100$, then $n(A^C \cup B^C)$ is equal to
(a) 260 (b) 560 (c) 360 (d) 600

38. If α, β are roots of the equation $p(x^2 - x) + x + 5 = 0$ and p_1, p_2 are two values of p for which the roots α, β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$ then the value of $\frac{p_1}{p_2}+\frac{p_2}{p_1}$ equals
(a) 254 (b) 0 (c) 245 (d) -254

39. The third term of a G.P. is 7, the product of its first five terms is
(a) 7^4 (b) 7^5 (c) 7^6 (d) 7^3

40. Four parts of 24 are in A.P. such that the ratio of product of extremes to product of means is 7 : 15, then four parts are
(a) $\frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \frac{21}{2}$ (b) $\frac{11}{2}, \frac{13}{2}, 3, 9$
(c) $\frac{5}{2}, \frac{15}{2}, \frac{9}{2}, \frac{21}{2}$ (d) $\frac{21}{2}, \frac{9}{2}, \frac{15}{2}, \frac{7}{2}$

41. If the coefficients of the r^{th} term and $(r + 1)^{th}$ term in the expansion of $(1 + x)^{20}$ are in the ratio 1 : 2, then r is equal to
(a) 6 (b) 7 (c) 8 (d) 9

42. The equation of tangents to the curve $f(x) = 1 + e^{-2x}$ where it cuts the line $y = 2$ is
(a) $x + 2y = 2$ (b) $2x + y = 2$
(c) $x - 2y = 1$ (d) $x - 2y + 2 = 0$

43. If the line $ax + by + c = 0$ is normal to the curve $xy + 5 = 0$, then
(a) $a > 0, b > 0$ (b) $b > 0, a < 0$
(c) $b < 0, a > 0$ (d) none of these

44. If $\log_2 x + \log_2 y \geq 6$, then the least value of $x + y$ is
(a) 4 (b) 8 (c) 16 (d) 32

45. The ratio in which the xy-plane divides the join of (1, 2, 3) and (4, 2, 1) is
(a) 3 : 1 internally (b) 3 : 1 externally
(c) 1 : 2 internally (d) 2 : 1 externally

46. If P is a point in space such that $OP = 12$ and $\overrightarrow{OP}$ is inclined at angles of 45° and 60° with OX and OY respectively, then the position vector of P is
(a) $6\hat{i}+6\hat{j} \pm 6\sqrt{2}\,\hat{k}$ (b) $6\hat{i}+6\sqrt{2}\,\hat{j} \pm 6\hat{k}$
(c) $6\sqrt{2}\,\hat{i}+6\hat{j} \pm 6\hat{k}$ (d) none of these

47. The point in which the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ meets the plane $x - 2y + z = 20$ is
(a) (7, –8, 26) (b) (8, 7, 26)
(c) (7, 8, 26) (d) none of these

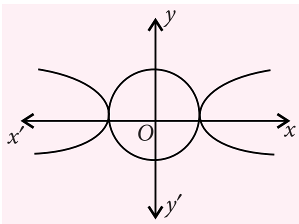
48. The equation of the plane perpendicular to the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$ and passing through the point (2, 3, 1) is
(a) $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})=1$
(b) $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=1$
(c) $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=7$
(d) none of these

49. The maximum value of $f(x) = |x \ln x|$ for $x \in (0, 1)$ is
(a) $1/e$ (b) e
(c) 1 (d) none of these

50. If $f(x)=\int_{x^2}^{x^3}\frac{dt}{\ln t}$, $x>0 \neq 1$ then
(a) $f(x)$ is an increasing function
(b) $f(x)$ has a minima at $x = 1$
(c) $f(x)$ is decreasing function
(d) $f(x)$ has a maxima at $x = 1$

SOLUTIONS

1. **(a) :** For the circle $x^2 + y^2 = 1$ and rectangular hyperbola $x^2 - y^2 = 1$, one common tangent is evidently $x = 1$, the other being $x = -1$.



We require the standard form of ellipse with focus at $S\left(\frac{1}{2}, 1\right)$ and directrix $x = 1$ which is

$$\left(x-\frac{1}{2}\right)^2+(y-1)^2=\left(\frac{1}{2}\right)^2(1-x)^2$$

$$\Rightarrow x^2-x+\frac{1}{4}+(y-1)^2=\frac{1}{4}(x^2-2x+1)$$

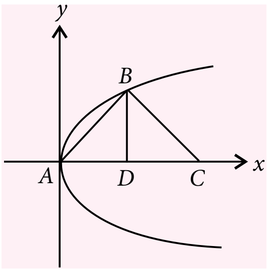
$$\Rightarrow \frac{3}{4}x^2-\frac{x}{2}+(y-1)^2=0$$

$$\Rightarrow \frac{3}{4}\left(x-\frac{1}{3}\right)^2+(y-1)^2=\frac{1}{12}$$

$$\Rightarrow 9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$$

2. (c): Let B be $(at^2, 2at)$

Slope of AB = $\frac{2at}{at^2}=\frac{2}{t}$



Equation of BC is

$$y-2at=-\frac{t}{2}(x-at^2)$$

$[\because BC \perp AB]$

Since BC meets x-axis at C.

$\therefore$ Ordinate of point C is zero.

$$0-2at=-\frac{t}{2}x+\frac{at^3}{2}$$

$4at = tx - at^3 \Rightarrow tx = 4at + at^3$

$\therefore$ $x = 4a + at^2$

Thus, coordinate of C is $(4a + at^2, 0)$ and D is $(at^2, 0)$.

$\therefore$ $DC = 4a + at^2 - at^2 = 4a$

3. (a) : The given equation is

$x^2 - 2ax + a^2 - 1 = 0$

$\Rightarrow$ $x^2 - (a + 1)x - (a - 1)x + (a + 1)(a - 1) = 0$

$\Rightarrow$ $x\{x - (a + 1)\} - (a - 1)\{x - (a + 1)\} = 0$

$\Rightarrow$ $(x - a + 1)(x - a - 1) = 0$

$\Rightarrow$ $x = a - 1$ or $x = a + 1$

Since, both the roots lie in $(-2, 1)$

$\therefore$ $-2 < a -1 < 1$ and $-2 < a + 1 < 1$

$\Rightarrow$ $-1 < a < 2$ and $-3 < a < 0$

$\therefore$ $-1 < a < 0 \Rightarrow [a] = -1$

4. (b) : Given $R = \{(x, y) : 2x + y = 8; x, y \in N\}$

Here $y = 8 - 2x$

$\therefore$ When $x = 1, y = 6 \in N$

$x = 2, y = 4 \in N$

$x = 3, y = 2 \in N$

$x = 4, y = 0 \notin N$

$\therefore$ $R = \{(1, 6), (2, 4), (3, 2)\}$

$\therefore$ Range of $R = \{y : (x, y) \in R\} = \{2, 4, 6\}$

5. (c): Here $f(x)=\frac{\sin 4\pi[x]}{1+[x]^2}$

$f(x) = 0$ [since sine of integral multiple of π is 0]

$\therefore$ $f'(x) = 0 \ \forall x$

6. (d) : $|z|$ is a positive real number.

$\therefore$ $\text{Arg}(-|z|) = \pi$

$$\therefore \int_0^{100} \pi\, dx = 100\pi$$

7. (b) : $\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}+\frac{n!}{3!(n-3)!}+\frac{n!}{5!(n-5)!}+....\right]$

$$=\frac{1}{n!}[{}^nC_1+{}^nC_3+{}^nC_5+....]$$

$$=\frac{1}{n!}\cdot 2^{n-1}$$

$[\because$ sum of the odd binomial coefficients is $2^{n-1}]$

8. (c): There should be exactly 4 correct and 4 wrong answers.

Let C stands for correct and W stands for wrong answers. One order of doing so is illustrated.

CCCCWWWW

The number of permutation of above mentioned demonstration is $\frac{8!}{4!\,4!}$.

Since each correct answer can be marked in 1 way and that for wrong in 3 ways.

$\therefore$ Required number of ways $=\frac{8!}{4!\,4!}\times 3^4=5670$

9. (b) : Let us mix the toys making $m + n$ number of distinct toys in all. Now Rina can pick up m toys out of these $m + n$ toys in ${}^{m+n}C_m$ ways which includes one way in which she can pick up her own original toys.

$\therefore$ Required number of ways = ${}^{m+n}C_m - 1$.

10. (c): Let the roots are α and β

$\therefore$ $\alpha + \beta = -2a$ and $\alpha\beta = b$

Also given that $|\alpha - \beta| \le 2m$

$\Rightarrow$ $(\alpha - \beta)^2 \le 4m^2$

Now, $(\alpha + \beta)^2 - 4\alpha\beta = (\alpha - \beta)^2$

$\Rightarrow$ $4a^2 - 4b \le 4m^2 \Rightarrow a^2 - b \le m^2$

$\Rightarrow$ $a^2 - m^2 \le b$ ... (i)

Also, the discriminant $4a^2 - 4b \ge 0$ (as roots are real)

$\Rightarrow$ $a^2 \ge b$ ... (ii)

From (i) and (ii), we have

$a^2 - m^2 \le b \le a^2$

i.e., $b \in [a^2 - m^2, a^2]$

11. (d) : We know that probability of any event lies between 0 and 1 and both 0 and 1 are inclusive.

$$\therefore \quad 0\le\frac{1+3p}{4}\le 1 \text{ gives } -\frac{1}{3}\le p\le 1$$

$$0\le\frac{1-p}{3}\le 1 \text{ gives } -2\le p\le 1$$

$$0 \le \frac{1-3p}{2} \le 1 \text{ gives } -\frac{1}{3} \le p \le \frac{1}{3}$$

Since the events are mutually exclusive.

$$\therefore \quad 0 \le \frac{1+3p}{4} + \frac{1-p}{3} + \frac{1-3p}{2} \le 1 \text{ gives } \frac{1}{13} \le p \le 1$$

The set of all values of p is $\frac{1}{13} \le p \le \frac{1}{3}$.

12. (a) : The given function

$$f(x) = \begin{cases} 3[x] - 5\dfrac{|x|}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$$

$$= \begin{cases} 3[x]-5 & \text{when } x > 0 \\ 3[x]+5 & \text{when } x < 0 \\ 2 & \text{when } x = 0 \end{cases}$$

Now, $\int\limits_{-3/2}^{2} f(x)dx$

$$= \int\limits_{-3/2}^{-1} f(x)dx + \int\limits_{-1}^{0} f(x)dx + \int\limits_{0}^{1} f(x)dx + \int\limits_{1}^{2} f(x)dx$$

$$= \int\limits_{-3/2}^{-1} -1dx + \int\limits_{-1}^{0} 2dx + \int\limits_{0}^{1} (-5)dx + \int\limits_{1}^{2} (-2)dx$$

$$= -1\left(-1+\frac{3}{2}\right) + 2(0+1) + (-5)(1-0) + (-2)(2-1)$$

$$= -1\left(\frac{1}{2}\right) + 2 - 5 - 2 = -\frac{1}{2} + 2 - 5 - 2 \;= -\frac{11}{2}$$

13. (c)

14. (d) : Let $y = f(x) \;\Rightarrow\; x = f^{-1}(y)$ .... (i)

$\therefore \quad y = 4x + 3 \quad \Rightarrow x = \dfrac{y-3}{4}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y-3}{4}$ [By using (i)]

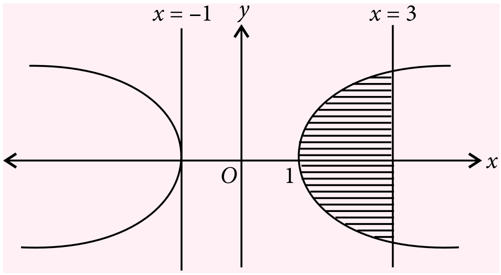
$\therefore \quad g(y) = \dfrac{y-3}{4}$

15. (c): The given equation of the curves

$|x - 1| \le 2$ and $x^2 - y^2 = 1$

$\therefore$ The graph of required curve is

$-1 \le x \le 3$, $x^2 - y^2 = 1$



$$\therefore \quad \text{Required area} = 2\int\limits_{1}^{3} \sqrt{x^2-1}\, dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left|x + \sqrt{x^2-1}\right|\right]_1^3$$

$$= \left[x\sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right|\right]_1^3$$

$$= 6\sqrt{2} - \log|3 + 2\sqrt{2}|$$

16. (d) : Let $P(r)$ be the probability that the number r is drawn in one draw. It is given that

$P(r) = kr$, where k is a constant.

Further, $P(1) + P(2) + ... + P(2n) = 1$

$\Rightarrow \; k(1 + 2 + 3 + ... + 2n) = 1$

$\Rightarrow \; k = \dfrac{1}{n(2n+1)}$

Hence, the required probability

$= P(2) + P(4) + P(6) + ... + P(2n)$

$= 2k(1 + 2 + ... + n)$

$= \dfrac{2}{n(2n+1)} \cdot \dfrac{n(n+1)}{2} = \dfrac{n+1}{2n+1}$

17. (a) : We know that $P(A\,|\,B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{P(A \cap B)}{P(A)} \cdot \frac{P(A)}{P(B)}$$

$$P(A\,|\,B) = P(B\,|\,A) \cdot \frac{P(A)}{P(B)}$$

$$\therefore \quad P(B) = \frac{\frac{2}{3} \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{3}$$

$$\left[\because \; P(B\,|\,A) = \frac{2}{3},\; P(A) = \frac{1}{4} \text{ and } P(A\,/\,B) = \frac{1}{2}\right]$$

18. (c): Let $|A| \ne 0$

$\therefore \quad A^{-1}$ exists.

Given that $AB = AC$

$\therefore \quad A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow$ $(A^{-1}A)B = (A^{-1}A)C$ [Associative law]
$\Rightarrow$ $B = C$ [$\because AA^{-1} = I$]
$\Rightarrow$ A is non-singular

19. (a) : Here $f(x)=\begin{vmatrix} \sin x & 1 & 0 \\ 1 & 2\sin x & 1 \\ 0 & 1 & 2\sin x \end{vmatrix}$

$= \sin x(4\sin^2 x - 1) - 1(2\sin x)$
$\Rightarrow$ $f(x) = 4\sin^3 x - 3\sin x$
$\therefore$ $f(-x) = -4\sin^3 x + 3\sin x = -(4\sin^3 x - 3\sin x)$
$\Rightarrow$ $f(-x) = -f(x)$
$\therefore$ $f(x)$ is an odd function.

$\therefore \int_{-\pi/2}^{\pi/2} f(x)dx = 0$

20. (b) : $f(x)=\dfrac{\sin x\cos x}{1+\sin x+\cos x}=\dfrac{\sin x}{\sec x+\tan x+1}$

$=\dfrac{\sin x}{(1+\tan x)+\sec x}=\dfrac{\sin x(1+\tan x-\sec x)}{(1+\tan x)^2-\sec^2 x}$

$=\dfrac{\cos x(1+\tan x-\sec x)}{2}=\dfrac{\sin x+\cos x-1}{2}$

21. (b) : $f(x)=e^{\cos^{-1}\sin\left(x+\frac{\pi}{3}\right)}$

$\therefore \quad f\left(\dfrac{8\pi}{9}\right)=e^{\cos^{-1}\sin\left(\frac{8\pi}{9}+\frac{\pi}{3}\right)}$

$=e^{\cos^{-1}\sin\left(\frac{11\pi}{9}\right)}=e^{\cos^{-1}\sin\left(\frac{22\pi}{18}\right)}$

$=e^{\cos^{-1}\sin\left(\frac{9\pi}{18}+\frac{13\pi}{18}\right)}=e^{\cos^{-1}\sin\left(\frac{\pi}{2}+\frac{13\pi}{18}\right)}$

$=e^{\cos^{-1}\cos\left(\frac{13\pi}{18}\right)}=e^{\frac{13\pi}{18}}$

22. (c): Given that $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ are in A.P.

$\Rightarrow \quad 2\sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}}=\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}$

$\Rightarrow$ $2(s - b)= (s - a) + (s - c)$
$\Rightarrow$ $2b = a + c$
$\therefore$ a, b, c are in A.P.

23. (d) : Given that $\omega=\dfrac{3z}{3z-i}$.

$\therefore \quad |\omega|=\dfrac{3|z|}{|3z-i|}$

$\Rightarrow$ $|3z - i| = 3|z|$
$\Rightarrow$ $|3(x) + i(3y - 1)| = |3(x + iy)|$ [$z = x + iy$]
$\Rightarrow$ $(3x)^2 + (3y - 1)^2 = 9(x^2 + y^2)$
$\Rightarrow$ $6y - 1 = 0$ which is a straight line.

24. (a) : $g(x)=\int x^x \log_e(ex)\,dx$

$=\int \dfrac{d}{dx}(x^x)dx$

$g(x) = x^x + C$
$\therefore$ $g'(x) = x^x \log(1 + x) = x^x \log_e(ex)$
$g'(\pi) = \pi^\pi \log_e(e\pi)$

25. (c): $xy = e - e^y$

$\Rightarrow \quad y+x\dfrac{dy}{dx}=-e^y\dfrac{dy}{dx}$

$\Rightarrow \quad e^y\dfrac{dy}{dx}+x\dfrac{dy}{dx}+y=0$ ... (i)

$\therefore \quad e\dfrac{dy}{dx}+1=0$ [As $x = 0$ $\therefore$ $y = 1$]

$\Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{1}{e}$

Differentiating (i) with respect to x, we get

$$e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2+x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Using $x = 0$, $y = 1$, $\dfrac{dy}{dx}=-\dfrac{1}{e}$, we get

$\dfrac{d^2y}{dx^2}=\dfrac{1}{e^2}$

26. (d) : Let $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$

$\therefore \quad |\vec{a}|^2=a_1^2+b_1^2+c_1^2$.

$\therefore \quad \vec{a}\times\hat{i}=-b_1\hat{k}+c_1\hat{j}$

$\Rightarrow \quad |\vec{a}\times\hat{i}|^2=b_1^2+c_1^2$.

Similarly, $|\vec{a}\times\hat{j}|^2=a_1^2+c_1^2$ and $|\vec{a}\times\hat{k}|^2=a_1^2+b_1^2$

$\therefore \quad |\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2(a_1^2+b_1^2+c_1^2)$

$=2|\vec{a}|^2$

27. (c): $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$=\begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$ $(C_3 \to C_1 + C_3)$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1$$

which is independent of x and y

28. (c): By L.M.V.T., $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{f(3)-f(1)}{3-1}$

$$f'(c) = \frac{\log_e 3 - \log_e 1}{2} = \frac{1}{2}\log_e 3$$

$$\Rightarrow \quad \frac{1}{c} = \frac{1}{2}\log_e 3 = \frac{1}{2\log_3 e}$$

$\therefore \quad c = 2\log_3 e$

29. (d) : $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right) + 1\right]$$

$$\therefore \quad v + x\frac{dv}{dx} = v[\log v + 1] \qquad \left[\text{Putting } \frac{y}{x} = v\right]$$

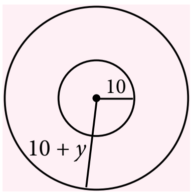
$$\Rightarrow \quad v + x\frac{dv}{dx} = v\log v + v$$

$$\Rightarrow \quad \int\frac{dv}{v\log v} = \int\frac{dx}{x} \quad \Rightarrow \quad \log\left(\frac{y}{x}\right) = Cx$$

30. (a) : $V = \dfrac{4}{3}\pi(y+10)^3$, where y is thickness of ice

$$\frac{dV}{dt} = 4\pi(10+y)^2\frac{dy}{dt}$$

$$\left(\frac{dy}{dt}\right)_{y=5} = \frac{50}{4\pi(15)^2}$$



$$\left[\text{As } \frac{dV}{dt} = 50 \text{ cm}^3/\text{min}\right]$$

$$= \frac{1}{18\pi} \text{ cm/min}$$

31. (c): $\displaystyle\int\frac{(\log x - 1)^2}{(1+(\log x)^2)^2}dx$

Let $\log x = z \quad \Rightarrow \quad e^z dz = dx$

$$= \int e^z\left(\frac{1}{1+z^2} - \frac{2z}{(1+z^2)^2}\right)dz$$

$$\therefore \quad \int e^z\left(f(z) + f'(z)\right)dz \qquad \left[f(z) = \frac{1}{1+z^2}\right]$$

$= e^z f(z) + C$

$$= \frac{e^z}{1+z^2} + C = \frac{e^{\log x}}{1+(\log x)^2} + C = \frac{x}{1+(\log x)^2} + C$$

32. (a) : Here $S_1 : x^2 + y^2 + 3x + 7y + 2p - 5 = 0$

and $S_2 : x^2 + y^2 + 2x + 2y - p^2 = 0$

$S_1 - S_2 = 0$

$\Rightarrow \quad x + 5y + 2p - 5 + p^2 = 0 \qquad \ldots \text{(i)}$

If there is a circle passing through P, Q and (1, 1), then its necessary and sufficient condition is that (1, 1) does not lie on PQ.

i.e. $1 + 5 + 2p - 5 + p^2 \neq 0$

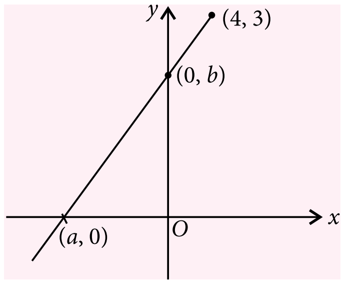
$\Rightarrow \quad p^2 + 2p + 1 \neq 0 \quad \Rightarrow \quad (p+1)^2 \neq 0$

$\therefore \quad p \neq -1$

Thus for all values of p except -1, there is a circle passing through P, Q and (1, 1).

33. (d) : Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots \text{(i)}$$



Given that $a + b = -1$

Since (i) is passing through (4, 3)

$$\therefore \quad \frac{4}{a} + \frac{3}{b} = 1$$

$$\Rightarrow \quad \frac{4}{a} + \frac{3}{[-(a+1)]} = 1$$

$\Rightarrow \quad 4a + 4 - 3a = a(a+1) \quad \Rightarrow \quad a^2 + a = a + 4$

$\Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = \pm 2$

$\therefore \quad$ When $a = 2$, $b = -3$ and $a = -2$, $b = 1$

So equations are $\dfrac{x}{2} - \dfrac{y}{3} = 1$ and $\dfrac{x}{-2} + \dfrac{y}{1} = 1$

34. (a) : We have, $\dfrac{x+4}{x-3} < 2$

$$\Rightarrow \quad \frac{x+4}{x-3} - 2 < 0 \quad \Rightarrow \quad \frac{-x+10}{x-3} < 0 \quad \Rightarrow \quad \frac{x-10}{x-3} > 0$$

$\Rightarrow \quad \{x - 10 > 0 \text{ and } x - 3 > 0\}$ or $\{x - 10 < 0 \text{ and } x - 3 < 0\}$

$\Rightarrow \quad \{x > 10 \text{ and } x > 3\}$ or $\{x < 10 \text{ and } x < 3\}$

$\Rightarrow \quad x \in (-\infty, 3) \cup (10, \infty)$

35. (d) : We have, $5x + 2 < 3x + 8$ and $\dfrac{x+2}{x-1} < 4$

$$\Rightarrow \quad x < 3 \text{ and } \frac{(x+2)(x-1)}{(x-1)^2} < 4, \ x \neq 1$$

$\Rightarrow$ $x < 3$ and $(x + 2)(x - 1) < 4x^2 - 8x + 4$
$\Rightarrow$ $x < 3$ and $3x^2 - 9x + 6 > 0$
$\Rightarrow$ $x < 3$ and $x^2 - 3x + 2 > 0$
$\Rightarrow$ $x < 3$ and $(x - 1)(x - 2) > 0$
$\Rightarrow$ $x < 3$ and $(x < 1$ or $x > 2) \Rightarrow x \in (-\infty, 1) \cup (2, 3)$

36. (d) : Let $P(n) = \frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$

For $n = 1$, $P(n)$ is not true.

For $n = 2$, $P(2): \frac{4^2}{2+1} < \frac{4!}{(2)^2} \Rightarrow \frac{16}{3} < \frac{24}{4}$

which is true.

Let for $n = m > 2$, $P(n)$ is true *i.e.* $\frac{4^m}{m+1} < \frac{(2m)!}{(m!)^2}$

Now, $\frac{4^{m+1}}{m+2} = \frac{4^m}{m+1} \cdot \frac{4(m+1)}{m+2} < \frac{(2m)!}{(m!)^2} \cdot \frac{4(m+1)}{m+2}$

$= \frac{(2m)!(2m+1)(2m+2) \times 4(m+1)(m+1)^2}{(2m+1)(2m+2)(m!)^2(m+1)^2(m+2)}$

$= \frac{[2(m+1)]!}{[(m+1)!]^2} \cdot \frac{2(m+1)^2}{(2m+1)(m+2)} < \frac{[2(m+1)]!}{[(m+1)!]^2}$

$\left[\because \frac{2(m+1)^2}{(2m+1)(m+2)} < 1 \ \forall m > 2\right]$

Hence, for $n \geq 2$, $P(n)$ is true.

37. (d) : We know, $n(A^C \cup B^C) = n[(A \cap B)^C]$
$= n(U) - n(A \cap B)$
$= n(U) - 100 = 700 - 100 = 600$

38. (a) : $p(x^2 - x) + x + 5 = 0 \Rightarrow px^2 - (p - 1)x + 5 = 0$

$\therefore$ $\alpha + \beta = \frac{p-1}{p}$ and $\alpha\beta = \frac{5}{p}$

Now, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$

$\Rightarrow \frac{(p-1)^2 - 10p}{5p} = \frac{4}{5} \Rightarrow p^2 - 16p + 1 = 0$

$\therefore$ $p_1 + p_2 = 16$, $p_1p_2 = 1$

Now, $\frac{p_1}{p_2} + \frac{p_2}{p_1} = \frac{(p_1+p_2)^2 - 2p_1p_2}{p_1p_2}$

$\therefore$ Required value is 254.

39. (b) : Given that $ar^2 = 7$ $\therefore$ $a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$
$= a^5 r^{1+2+3+4} = a^5 r^{10} = (ar^2)^5 = 7^5$

40. (a) : Let the numbers are
$a - 3d, a - d, a + d, a + 3d$

$\therefore$ $4a = 24 \Rightarrow a = 6$

Again, $\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15} \Rightarrow d = \pm\frac{3}{2}$

$\therefore$ Required numbers are $\frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \frac{21}{2}$

41. (b) : $t_r = t_{(r-1)+1} = {}^{20}C_{r-1}\, x^{r-1}$ and $t_{r+1} = {}^{20}C_r\, x^r$

According to the problem,

$\frac{{}^{20}C_{r-1}}{{}^{20}C_r} = \frac{1}{2} \Rightarrow \frac{r}{20-r+1} = \frac{1}{2}$

$\Rightarrow 2r = 20 - r + 1 \Rightarrow 3r = 21$
$\therefore$ $r = 7$

42. (b) : When $y = 2$, then $e^{-2x} = 1 \Rightarrow x = 0$

Now, $\frac{df}{dx} = -2e^{-2x}$ $\therefore$ $\left.\frac{df}{dx}\right|_{x=0} = -2$

$\therefore$ The equation of the tangent is
$y - 2 = -2(x - 0) \Rightarrow 2x + y - 2 = 0$

43. (a) : $xy = -5 \Rightarrow x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$[\because xy = -5]$

Since, the line $ax + by + c = 0$ is normal to the curve

$\therefore$ $\frac{-y}{x} \times \frac{-a}{b} = -1 \Rightarrow \frac{-a}{b} < 0$

$\Rightarrow$ the slope of the normal is negative

$\Rightarrow -\frac{a}{b} < 0 \Rightarrow \frac{a}{b} > 0 \Rightarrow a > 0, b > 0$ or $a < 0, b < 0$

44. (c): Given that $\log_2 x + \log_2 y \geq 6 \Rightarrow \log_2(xy) \geq 6$
$\Rightarrow xy \geq 64$
Also for $\log_2 x$ and $\log_2 y$ to be defined, $x > 0, y > 0$
$\because$ AM $\geq$ GM,

$\therefore$ $\frac{x+y}{2} \geq \sqrt{xy} \Rightarrow x + y \geq 2\sqrt{xy} \geq 2\sqrt{64} = 16$

45. (b) : Suppose xy-plane divides the join of (1, 2, 3) and (4, 2, 1) in the ratio $\lambda : 1$. Then the coordinates of the point of division are

$\left(\frac{4\lambda+1}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{\lambda+3}{\lambda+1}\right)$

The point lies on xy-plane. So, z-coordinate = 0

$\Rightarrow \frac{\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = -3$

Hence, xy – plane divides the join of (1, 2, 3) and (4, 2, 1) externally in the ratio 3 : 1.

46. (c): Let l, m, n be the direction cosines of $\overrightarrow{OP}$.

Then $l = \cos 45° = \frac{1}{\sqrt{2}}$, $m = \cos 60° = \frac{1}{2}$

Now, $l^2 + m^2 + n^2 = 1 \quad \Rightarrow \quad n = \pm\dfrac{1}{2}$

Let $\overrightarrow{OP} = \vec{r}$

$$\therefore \quad \vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k}) = 12\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{2}\hat{k}\right)$$

$$= 6\sqrt{2}\,\hat{i} + 6\hat{j} \pm 6\hat{k}$$

47. (b) : Any point on line is $(3r + 2, 4r - 1, 12r + 2)$ and if it lies in the plane also, then

$3r + 2 - 8r + 2 + 12r + 2 - 20 = 0$

$\Rightarrow 7r = 14 \quad \Rightarrow \quad r = 2$

Hence the required point is $(8, 7, 26)$

48. (b) : The given line is parallel to the vector $\vec{x} = \hat{i} - \hat{j} + 2\hat{k}$.

The required plane passes through the point $(2, 3, 1)$ *i.e.* $2\hat{i} + 3\hat{j} + \hat{k}$ and is perpendicular to the vector $\vec{n} = \hat{i} - \hat{j} + 2\hat{k}$.

So, its equation is

$$[\vec{r} - (2\hat{i} + 3\hat{j} + \hat{k})]\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 1$$

49. (a) : $f(x) = |x \ln x|$

For $x \in (0, 1)$, $f(x) = -x \ln x$

$$f'(x) = -\left(x\cdot\frac{1}{x} + \ln x\right) = -(1 + \ln x)$$

$$f'(x) = 0 \Rightarrow x = \frac{1}{e},\quad f''(x) = -\frac{1}{x} < 0.$$

$f(x)$ will be maximum at $x = 1/e$.

Maximum value of $f(x) = -\dfrac{1}{e}\ln\dfrac{1}{e} = \dfrac{1}{e}$.

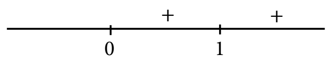
50. (a) : $f(x) = \displaystyle\int_{x^2}^{x^3} \frac{dt}{\ln t}$

For increasing or decreasing function,

$$f'(x) = \frac{1}{\ln x^3}\cdot 3x^2 - \frac{1}{\ln x^2}\cdot 2x$$

$$= \frac{1}{\ln x}(x^2 - x)$$

Sign of $f'(x)$



Since $f'(x) > 0$ for $x > 0$, $x \neq 1$

Hence $f(x)$ is increasing function.

It does not have minima as $x = 0$ is not in its domain.

Contd. from page no. 30

JEE WORK CUTS

40. If $f(x) = A\sin\left(\dfrac{\pi x}{2}\right) + B$, $f'\left(\dfrac{1}{2}\right) = \sqrt{2}$ and $\displaystyle\int_0^1 f(x)dx = \frac{2A}{\pi}$. Then the constants A and B are, respectively

(a) $\pi/2$ and $\pi/2$ (b) $2/\pi$ and $3/\pi$

(c) 0 and $-4/\pi$ (d) $4/\pi$ and 0

41. $\displaystyle\lim_{x\to\infty} \frac{\log[x]}{x}$, where $[x]$ is the greatest integer less than or equal to x is

(a) 1 (b) -1

(c) does not exist (d) 0

42. If at any point (x, y) on a curve, subtangent and subnormal are of equal length, then the length of tangent is

(a) $\sqrt{2y}$ (b) $|\sqrt{2}y|$

(c) $|y|$ (d) None of these

43. If a straight line in space is equally inclined to the co–ordinate axes, the cosine of its angle of inclination to any one of the axes is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{\sqrt{3}}$ (d) $\dfrac{1}{\sqrt{2}}$

44. Let $\vec{v} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{w} = \hat{i} + 3\hat{k}$. If $\vec{u}$ is a unit vector, then the maximum value of the scalar triple product $[\vec{u}\,\vec{v}\,\vec{w}]$ is

(a) -1 (b) $\sqrt{10} + \sqrt{6}$

(c) $\sqrt{59}$ (d) $\sqrt{60}$

ANSWER KEYS

1.	(b)	**2.**	(b)	**3.**	(c)	**4.**	(a)	**5.**	(a)
6.	(d)	**7.**	(b)	**8.**	(b)	**9.**	(b)	**10.**	(c)
11.	(a)	**12.**	(b)	**13.**	(d)	**14.**	(a)	**15.**	(c)
16.	(b)	**17.**	(b)	**18.**	(a)	**19.**	(a)	**20.**	(b)
21.	(b)	**22.**	(b)	**23.**	(a)	**24.**	(a)	**25.**	(b)
26.	(c)	**27.**	(a)	**28.**	(b)	**29.**	(a)	**30.**	(c)
31.	(a)	**32.**	(a)	**33.**	(b)	**34.**	(b)	**35.**	(a)
36.	(a)	**37.**	(a)	**38.**	(a)	**39.**	(c)	**40.**	(d)
41.	(d)	**42.**	(b)	**43.**	(c)	**44.**	(c)		

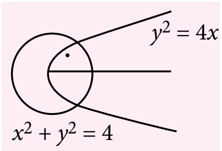
For detailed solution to the Sample Paper, visit our website : www. vidyalankar.org.

MATHS MUSING

SOLUTION SET-157

1. **(a) :** Differentiating $h(x) = \{f(x)\}^2 + \{g(x)\}^2$ w.r.t. x,
$h'(x) = 2f(x)\cdot f'(x) + 2g(x)\cdot g'(x)$ ...(1)
Now, $f'(x) = g(x) \Rightarrow f''(x) = g'(x)$
$\Rightarrow \quad -f(x) = g'(x) \qquad [\because f''(x) = -f(x)]$
$\therefore \quad (1) \Rightarrow h'(x) = 2f(x)\cdot g(x) + 2g(x)\cdot\{-f(x)\} = 0$
$\therefore \quad h(x) = c$; but $h(5) = 11$. So $11 = c$.
$\therefore \quad h(x) = 11 \ \forall x$
$\therefore \quad h(10) = 11.$

2. **(b) :** The point $P(-2a, a+1)$ will be an interior point of both the circle $x^2 + y^2 - 4 = 0$ and the parabola $y^2 - 4x = 0$



$(-2a)^2 + (a+1)^2 - 4 < 0$
i.e., $5a^2 + 2a - 3 < 0$ ...(1)
and $(a+1)^2 - 4(-2a) < 0$,
i.e., $a^2 + 10a + 1 < 0$ ...(2)
From (1), $(5a-3)(a+1) < 0$
$\therefore$ By sign-scheme we get, $-1 < a < \frac{3}{5}$ ...(3)
By sign-scheme for (2), we get
$-5 - 2\sqrt{6} < a < -5 + 2\sqrt{6}$ ...(4)
The set of values of a safisfying (3) and (4) is
$-1 < a < -5 + 2\sqrt{6}$

3. **(a) :** The new equation is $2x - y + 3z + 5 + \lambda(5x - 4y - 2z + 1) = 0$
$(2 + 5\lambda)x - (1 + 4\lambda)y + (3 - 2\lambda)z + 5 + \lambda = 0$ ...(1)
It is perpendicular to $2x - y + 3z + 5 = 0$.
$\therefore \quad 2(2 + 5\lambda) + (1 + 4\lambda) + 3(3 - 2\lambda) = 0$
$\Rightarrow \quad 14 + 8\lambda = 0 \Rightarrow \lambda = -\frac{7}{4}$
Now, from (1), the required equation of plane is
$\Rightarrow \quad 27x - 24y - 26z = 13.$

4. **(d):** $S = (1+x)^{100} + 2x(1+x)^{99} + 3x^2(1+x)^{98} + \ldots\ldots + 101x^{100}$

$$\frac{Sx}{1+x} = (1+x)^{100}\left[\frac{x}{1+x} + 2\left(\frac{x}{1+x}\right)^2 + 3\left(\frac{x}{1+x}\right)^3 + \ldots + 101\left(\frac{x}{1+x}\right)^{101}\right]$$

Subtraction of this from the above,

$$\frac{S}{1+x} = (1+x)^{100}\left[(1+x)\left(1 - \left(\frac{x}{1+x}\right)^{101}\right) - 101\left(\frac{x}{1+x}\right)^{101}\right]$$

$\therefore \quad S = (1+x)^{102} - (1+x)x - 101x^{101}$. The coefficient of x^{50} is $\binom{102}{50}$

5. **(b) :** Taking the line as $x = -5 + r\cos\theta$,
$y = -4 + r\sin\theta$. It meets $x + 3y + 2 = 0$ at B.
$\therefore \quad -5 + r\cos\theta + 3(-4 + r\sin\theta) + 2 = 0$
$\therefore \quad \frac{15}{AB} = \cos\theta + 3\sin\theta$
Likewise, $\frac{10}{AC} = 2\cos\theta + \sin\theta$ and $\frac{6}{AD} = \cos\theta - \sin\theta$
$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2 \Rightarrow 3\sin\theta + 2\cos\theta = 0$
or $\tan\theta = -\frac{2}{3}$
$\therefore$ The line is $y + 4 = -\frac{2}{3}(x+5)$ or $2x + 3y + 22 = 0$.

6. **(a, c) :** Let S be the sample space, then
$n(S) = 11 \times 11 = 121$
Consider,
$A = \{(x, y): |x - y| = 0\} = \{(0, 0), \ldots (10, 10)\}$
$\Rightarrow n(A) = 11$
$B = \{(x, y): |x - y| = 1\} \Rightarrow n(B) = 20$
$C = \{(x, y): |x - y| = 2\} \Rightarrow n(C) = 18$
$D = \{(x, y): |x - y| = 3\} \Rightarrow n(D) = 16$
$E = \{(x, y): |x - y| = 4\} \Rightarrow n(E) = 14$
$F = \{(x, y): |x - y| = 5\} \Rightarrow n(F) = 12$
(i) At random case: Probability
$$= \frac{n(A) + n(B) + n(C) + n(D) + n(E) + n(F)}{n(S)} = \frac{91}{121}$$

Solution Sender of Maths Musing

SET-157

1. Gajula Ravinder (Karim Nagar)
2. Khokon Kumar Nandi (West Bengal)
3. N.Jayanthi (Hyderabad)
4. Gouri Sankar Adhikari (West Bengal)

SET-156

1. Gouri Sankar Adhikari (West Bengal)
2. Gajula Ravinder (Karim Nagar)
3. Khokon Kumar Nandi (West Bengal)
4. Divyesh Saglani (Hyderabad)

(ii) Without random case: Probability

$$=\frac{n(B)+n(C)+n(D)+n(E)+n(F)}{n(S)}=\frac{80}{21}$$

7. (d): $G_n=((n+1)(n+2)...(n+n))^{\frac{1}{n}}$

$$\frac{G_n}{n}=\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

$$\therefore\quad \lim_{n\to\infty}\ln\left(\frac{G_n}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\ln(1+x)dx=2\ln 2-1=\ln\left(\frac{4}{e}\right)$$

$$\left[\because\ \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx\right]$$

$$\therefore\quad \lim_{n\to\infty}\frac{G_n}{n}=\frac{4}{e}$$

8. (c) : $R_n=\left(\dfrac{(n+1)^2+(n+2)^2+...+(n+n)^2}{n}\right)^{\frac{1}{2}}$

$$\lim_{n\to\infty}\left(\frac{R_n}{n}\right)^2=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(1+\frac{r}{n}\right)^2=\int_0^1(1+x)^2dx=\frac{7}{3}$$

$$\therefore\quad \lim_{n\to\infty}\frac{R_n}{n}=\sqrt{\frac{7}{3}}$$

9. (9) : $\displaystyle\sum_{i=0}^{n}\sum_{j=1}^{n}{}^nC_j\,{}^jC_i$

$= {}^nC_1({}^1C_0+{}^1C_1)+{}^nC_2({}^2C_0+{}^2C_1+{}^2C_2)$
$+{}^nC_3({}^3C_0+{}^3C_2+{}^3C_3)+{}^nC_4({}^4C_0+{}^4C_1+{}^4C_2+{}^4C_4)$
$+ ... + {}^nC_n({}^nC_0+{}^nC_1+{}^nC_2+...+{}^nC_n)$
$= {}^nC_1(2)+{}^nC_2(2)^2+{}^nC_3(2)^3+{}^nC_4(2)^4+...+{}^nC_n(2)^n$
$= (1+2)^n-1=3^n-1$

10. (b): P. Since $a>0$, $b>0$ and $c>0$

$$\Rightarrow\quad \frac{(b+c)\log_e a}{b^2-c^2}=\frac{(c+a)\log_e b}{c^2-a^2}=\frac{(a+b)\log_e c}{a^2-b^2}$$

$$=\frac{(b+c)\log_e a+(c+a)\log_e b+(a+b)\log_e c}{0}$$

$\therefore\quad \log_e a^{b+c}+\log_e b^{c+a}+\log_e c^{a+b}=0$

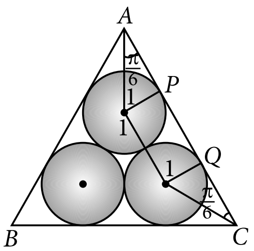
$\Rightarrow\quad \log_e(a^{b+c}+b^{c+a}+c^{a+b})=0$

$\Rightarrow\quad a^{b+c}+b^{c+a}+c^{a+b}=1$

Q. $AC=AP+PQ+QC$

$$=1\cdot\cot\frac{\pi}{6}+2+1\cdot\cot\frac{\pi}{6}=2(1+\sqrt{3})$$

$$ar(\Delta ABC)=\frac{\sqrt{3}}{4}AC^2=\frac{\sqrt{3}}{4}\cdot4(1+\sqrt{3})^2$$
$$=6+4\sqrt{3}.$$



R. $8^{2n}-(62)^{2n+1}=(9-1)^{2n}-(63-1)^{2n+1}$
$=$ (multiple of 9) $+ 1 -$ ((multiple of 9) -1)
$=$ multiple of 9 + 2.

S. $\displaystyle\int e^{-2x}(3\cos 5x-4\sin 5x)dx$

$$=e^{-2x}(a\cos 5x+b\sin 5x)+c$$

$$\Rightarrow\quad e^{-2x}\left(\frac{14}{29}\cos 5x+\frac{23}{29}\sin 5x\right)$$
$$=e^{-2x}(a\cos 5x+b\sin 5x)$$

On comparing, we get a and b.

$$\therefore\quad a+b=\frac{14}{29}+\frac{23}{29}=\frac{37}{29}$$

PRACTICE PAPER

JEE ADVANCED

SECTION - I

This section contains 25 multiple choice questions. Each question has four choices (a), (b), (c) and (d), out of which only one is correct.

Marks: $25 \times 3 = 75$ Negative marking (-1)

1. The number of solutions of $\frac{5+3x}{6}+\frac{1+x}{2}<\frac{19}{3}$, x belongs to the set of whole numbers, is
(a) 3 (b) 4
(c) 5 (d) none of these

2. If $-3 \le a \le 5$, $-5 \le b \le -2$, then
(a) $0 \le a^2 + b^2 \le 25$ (b) $0 \le a^2 + b^2 \le 50$
(c) $4 \le a^2 + b^2 \le 50$ (d) $13 \le a^2 + b^2 \le 50$

3. The number of integer values of $30x$, where x belongs to the solution set of $0<\left|3-\frac{1}{x}\right|<\frac{1}{2}$, is
(a) 2 (b) 3
(c) 4 (d) 5

4. Total number of relations from A to B is 4096, the number of relations from B to C is 64 and the number of relations from A to C is 256 .Then which of the following is definitely true ?
(a) $n(A) = 6$ (b) $n(B) = 4$
(c) $n(C) = 2$ (d) $n(A) = 3$

5. The number of integers in the solution set of the inequation $12 \le |4x - 17| \le 19$, are
(a) 2 (b) 3
(c) 0 (d) 4

6. $P(n)$: $n^2 - n + 2$ is divisible by 11. This statement is true
(a) for all $n \ge 5$ (b) for all $n \ge 7$
(c) for $n = 5, 6, 7$ (d) none of these

7. If $z = 1 + i\tan\alpha$, $\pi < \alpha < \frac{3\pi}{2}$, then modulus of $-5iz$ is
(a) $5\cos\alpha$ (b) $\frac{-5}{\cos\alpha}$
(c) $\frac{5}{\cos\alpha}$ (d) $5\cot\alpha$

8. $(5i^{26} + 3i^{39})^2 =$
(a) $16 + 30i$ (b) $34 + 30i$
(c) $16 - 30i$ (d) $34 - 30i$

9. If $z=\frac{18}{-1+\sqrt{3}i}$, then principal arg$(iz)$ =
(a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $-\frac{2\pi}{3}$ (d) $\frac{2\pi}{3}$

10. $P(n)$: $3^n > 9n + 2$. This statement is true
(a) for all $n \in N$ (b) for all $n \ge 2$
(c) for all $n \ge 4$ (d) none of these

11. Three sets A, B and C are such that $A = B \cap C$ and $B = C \cap A$, then
(a) $A \subset B$ (b) $A \supset B$
(c) $A = B$ (d) $A \subset B'$

12. $A = \{x : x$ is a prime factor of 240$\}$, $B = \{x : x$ is the sum of any two prime factors of 240$\}$. Then
(a) $A = B$ (b) $7 \in A \cap B$
(c) $8 \in A \cap B$ (d) $n(A \cap B) = 1$

13. A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product P_1 and 1450 consumers liked product P_2. The least number of consumers who must have liked both the products is
(a) 1170 (b) 3170
(c) 270 (d) none of these

14. If X and Y are two sets, then $X \cap (Y \cup X)'$ equals
(a) X (b) Y
(c) ϕ (d) none of these

15. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b): |a^2 - b^2| > 16\}$, then
(a) Domain = $\{5\}$ (b) Domain = $\{4, 5\}$
(c) Domain = $\{1, 2, 5\}$ (d) none of these

16. If $z=\left(-\frac{1}{\sqrt{3}}+\frac{1}{2}i\right)^5+\left(-\frac{1}{\sqrt{3}}-\frac{1}{2}i\right)^5$, then
(a) Re $(z) = 0$ (b) Im $(z) = 0$
(c) Re $(z) > 0$, Im $(z) < 0$
(d) Re $(z) < 0$, Im $(z) > 0$

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17. If A and B are two sets such that $n(A \cap \overline{B}) = 5$, $n(\overline{A} \cap B) = 4$ and $n(A \cup B) = 12$, then
(a) $n(A \cap B) = 6$ (b) $n(A \cap B) = 2$
(c) $n(A \times B) = 72$ (d) $n(A \times B) = 56$

18. Domain of $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is
(a) $[0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, 0]$ (d) $(0, \infty)$

19. If $3z + 2\overline{z} = 5 + 2i$, then
(a) $\text{Re}(z) = 2$ (b) $\text{Im}(z) = 2i$
(c) $|z| = 5$ (d) none of these

20. If $(a + ib)(c + id) = 3 + 5i$, then the value of $(a^2 + b^2)(c^2 + d^2)$ is
(a) $-4 + 30i$ (b) $4 - 30i$
(c) 29 (d) 34

21. If $|z| = z + 3 - 2i$, then z equals
(a) $\dfrac{7}{6} + i$ (b) $-\dfrac{7}{6} + 2i$
(c) $-\dfrac{5}{6} + 2i$ (d) $\dfrac{5}{6} + i$

22. If z is multiplicative inverse of $5 + 12i$, then $\text{Re}(13z) + \text{Im}(13z)$ is equal to
(a) $\dfrac{13}{5 + 12i}$ (b) $65 + 156i$
(c) $-\dfrac{7}{13}$ (d) -7

23. If z, w are complex numbers and $|3z| = 5$, then $\left|\dfrac{75 - 27z\overline{w}}{z - w}\right| =$
(a) 12 (b) 15
(c) 45 (d) none of these

24. If $A = \{z : (1 + 2i)\overline{z} + (1 - 2i)z + 2 = 0\}$ and $B = \{z : (3 + 2i)\overline{z} + (3 - 2i)z + 3 = 0\}$, then
(a) $n(A \cap B) = 1$ (b) $A \subseteq B$
(c) $B \subseteq A$ (d) $A \cap B = \phi$

25. If z_1 and z_2 are roots of $z^2 - z + 1 - i = 0$, then a possible value of $\dfrac{z_1}{z_2}$ is
(a) $1 + i$ (b) $2 + 2i$
(c) $-1 - i$ (d) none of these

SECTION-II

This section contains 5 multiple choice questions. Each question has four choices (a), (b), (c) and (d), out of which one or more than one is correct.

Marks: $5 \times 4 = 20$ Negative marking (-2)

26. Let A and B have 3 and 6 elements respectively, then which of the following is correct.
(a) minimum value of $n(A \cap B)$ is 3
(b) maximum value of $n(A \cap B)$ is 3
(c) minimum value of $n(A \cup B)$ is 6
(d) maximum value of $n(A \cup B)$ is 9

27. S is a relation defined on R and is given by $(x, y) \in S$ if $|x - y| \le 1$, then ...
(a) $(x, x) \in S \ \forall\ x \in R$
(b) $(x, y) \in S$ implies $(y, x) \in S$
(c) $(x, x^2) \in S \ \forall\ x \in [-1, 1]$
(d) $(x, y) \in S$, $(y, z) \in S$ implies $(x, z) \in S$

28. If $|2z + 5| = 6z - 9$ and $|z|^2 = k\,\text{Re}(z) + \gamma$, where k and γ are real numbers, then...
(a) $k - 8\gamma = 18$ (b) $k + 4\gamma + 3 = 0$
(c) $k = 4$ (d) $\gamma = -7$

29. The relations f and g are defined by
$$f(x) = \begin{cases} 2ax^2 + b, 0 \le x \le 3 \\ 4x, 3 \le x \le 10 \end{cases}$$
$$g(x) = \begin{cases} x^2, 0 \le x \le 2 \\ ax + b, 2 \le x \le 10 \end{cases}$$
. If these relations are also functions then
(a) $6a + b = 10$ (b) $18a + b = 12$
(c) $2a + b = 4$ (d) $10a + b = 8$

30. If $-4 \le x < 5$ then
(a) Range of $f(x) = x^2 + 6x + 10$ is $[1, 65)$
(b) Range of $-\dfrac{2}{x+3}$ is $\left(-\infty, -\dfrac{1}{4}\right) \cup [2, \infty)$
(c) Range of $5 - |x + 3|$ is $(-3, 0]$
(d) Range of $-\dfrac{2}{x^2 + 10x + 27}$ is $\left[-\dfrac{2}{3}, -\dfrac{1}{51}\right)$

SOLUTIONS

1. **(c) :** Multiplying both sides by 6 and simplifying, we get $x < 5$.

2. **(c) :** $-3 \le a \le 5 \Rightarrow 0 \le a^2 \le 25$ and $-5 \le b \le -2$
$\Rightarrow 4 \le b^2 \le 25 \Rightarrow 4 \le a^2 + b^2 \le 50$

3. **(a) :** $0 < \left|3 - \dfrac{1}{x}\right| < \dfrac{1}{2} \Rightarrow -\dfrac{1}{2} < 3 - \dfrac{1}{x} < \dfrac{1}{2}, 3 - \dfrac{1}{x} \ne 0$
$$\Rightarrow \frac{5}{2} < \frac{1}{x} < \frac{7}{2}, x \ne \frac{1}{3}$$
$$\Rightarrow \frac{2}{7} < x < \frac{2}{5}, x \ne \frac{1}{3} \Rightarrow \frac{60}{7} < 30x < 12, 30x \ne 10$$
$\Rightarrow$ $30x = 9, 11$

4. **(c) :** Let $n(A) = p, n(B) = q, n(C) = r$, then $2^{pq} = 4096 = 2^{12}$
$\Rightarrow$ $pq = 12$
Similarly $qr = 6$ and $pr = 8$
$\Rightarrow$ $(pqr)^2 = 12 \times 6 \times 8 \Rightarrow pqr = 24$

5. **(d):** $12 \le |4x - 17| \le 19$
$\Rightarrow$ $12 \le 4x - 17 \le 19$ or $-19 \le 4x - 17 \le -12$
$\Rightarrow$ $\frac{29}{4} \le x \le 9$ or $-\frac{1}{2} \le x \le \frac{5}{4}$

6. **(d)**

7. **(b):** $|1 + i\tan\alpha| = \sqrt{1+\tan^2\alpha} = |\sec\alpha|$.
Now $\pi < \alpha < \frac{3\pi}{2} \Rightarrow |\sec\alpha| = -\sec\alpha$
$\Rightarrow$ $|-5iz| = 5|z| = -5\sec\alpha$

8. **(a) :** $i^{26} = -1, i^{39} = -i$
$\therefore (5i^{26} + 3i^{39})^2 = (-5 - 3i)^2$
$= 25 - 9 + 30i = 16 + 30i$

9. **(b):** $z = \frac{18}{-1+\sqrt{3}i}$
$\Rightarrow$ $iz = \frac{18i}{-1+\sqrt{3}i} \Rightarrow iz = \frac{18i(-1-\sqrt{3}i)}{4} = \frac{18}{4}(\sqrt{3} - i)$
$\Rightarrow$ $\arg(iz) = -\frac{\pi}{6}$

10. **(c)**

11. **(c)**

12. **(d):** $240 = 2^4.3.5 \Rightarrow A = \{2, 3, 5\}$ and $B = \{5, 7, 8\}$

13. **(a) :** $|U| = 2000, |P_1| = 1720, |P_2| = 1450,$
$|P_1 + P_2| = |P_1| + |P_2| - |P_1P_2| \Rightarrow |P_1 + P_2| = 3170 - |P_1P_2|$
but $|P_1 + P_2| \le 2000 \Rightarrow 3170 - |P_1P_2| \le 2000$
$\Rightarrow$ $|P_1P_2| \ge 1170$

14. **(c)**

15. **(c) :** $R = \{(1, 5), (2, 5), (5, 1), (5, 2)\}$
$\therefore$ Domain $= \{1, 2, 5\}$

16. **(b):** Here $z = \bar{z} \Rightarrow \text{Im}(z) = 0$

17. **(d):** $|AB| = |A + B| - |A\bar{B}| - |\bar{A}B| = 3$
Also $|A| = |A\bar{B}| + |AB| = 8$
Similarly $|B| = 7$

18. **(b):** $f(x)$ is defined if $|x| - x > 0$ *i.e.* $|x| > x \Rightarrow x < 0$

19. **(d):** $z = a + ib \Rightarrow a = 1, b = 2$
$\therefore$ $\text{Re}(z) = 1, \text{Im}(z) = 2$ and $|z| = \sqrt{5}$

20. **(d)**

21. **(c) :** $z = a + ib \Rightarrow \sqrt{a^2 + b^2} = a + ib + 3 - 2i$
$\Rightarrow$ $b = 2, a = -\frac{5}{6}$

22. **(c) :** $z = \frac{1}{5+12i} = \frac{5-12i}{169} \Rightarrow 13z = \frac{5-12i}{13}$

23. **(c) :** $|3z| = 5 \Rightarrow |3z|^2 = 25 \Rightarrow 9z\bar{z} = 25$
Now, $\left|\frac{75 - 27z\bar{w}}{z - w}\right| = \left|\frac{27z\bar{z} - 27z\bar{w}}{z - w}\right|$
$= 27\left|\frac{z(\bar{z} - \bar{w})}{z - w}\right| = 27|z|\left|\frac{\overline{z - w}}{z - w}\right| = 9|3z| = 45$

24. **(a):** $(1 + 2i)\bar{z} + (1 - 2i)z + 2 = 0 \Rightarrow x + 2y + 1 = 0$
and $(3 + 2i)\bar{z} + (3 - 2i)z + 3 = 0 \Rightarrow 6x + 4y + 3 = 0$
$\Rightarrow$ A and B have one common element

25. **(d):** $z^2 - z + 1 - i = 0 \Rightarrow z = \frac{1 \pm \sqrt{1 - 4(1-i)}}{2}$
$\Rightarrow$ $z = 1 + i, -i \Rightarrow \frac{z_1}{z_2} = \frac{-i}{1+i} = -\frac{i(1-i)}{2}$
or $\frac{z_1}{z_2} = \frac{1+i}{-i} = -1 + i$

26. **(b, c, d) :** $0 \le n(A \cap B) \le 3$ and $6 \le n(A \cup B) \le 9$

27. **(a, b)**

28. **(a, b, c) :** $|2z + 5| = |6z - 9| \Rightarrow |2z + 5|^2 = |6z - 9|^2$
$\Rightarrow$ $4|z|^2 + 10(z + \bar{z}) + 25 = 9[4|z|^2 + 9 - 6(z + \bar{z})]$
$\Rightarrow$ $32|z|^2 = 64(z + \bar{z}) - 56 \Rightarrow |z|^2 = 2(z + \bar{z}) - \frac{7}{4}$
$\Rightarrow$ $|z|^2 = 4\text{Re}(z) - \frac{7}{4}$

29. **(b, c, d)**

30. **(a, b, d) :** $-4 \le x < 5 \Rightarrow -1 \le x + 3 < 8 \Rightarrow 0 \le (x + 3)^2 < 64$
$\Rightarrow$ $1 \le (x + 3)^2 + 1 < 65 \Rightarrow 1 \le f(x) < 65$
$-4 \le x < 5$
$\Rightarrow$ $-1 \le x + 3 < 8 \Rightarrow -1 \le x + 3 < 0$ or $0 < x + 3 < 8$
$\Rightarrow$ $-\infty < \frac{2}{x+3} \le -2$ or $\frac{1}{4} < \frac{2}{x+3} < \infty$
$\Rightarrow$ $2 \le f(x) < \infty$ or $-\infty < f(x) < -\frac{1}{4}$
$-4 \le x < 5 \Rightarrow 1 \le x + 5 < 10 \Rightarrow 1 \le (x + 5)^2 < 100$
$\Rightarrow$ $3 \le (x + 5)^2 + 2 < 102 \Rightarrow -\frac{2}{3} \le f(x) < -\frac{1}{51}$

Rvp